Math 55b: Honors Real and Complex Analysis

Homework Assignment #5 (23 February 2017):
Univariate differential and integral calculus

Bitte ve[r]giß alles, was Du auf der Schule gelernt hast;
denn Du hast is nicht gelernt. *Emil Landau*[1]

Admittedly that's a bit extreme, but it is true that for many of you Stieltjes integrals, especially of vector-valued functions, are a new path in the familiar territory of integration, and might require a different kind of thinking. The second part of this problem set is geared towards developing such "new kinds of thinking", including a somewhat open-ended problem or two to suggest further directions in analysis that we won't pursue in Math 55.

But first, more about derivatives and Taylor series. We start one of the few cases where the theory developed so far lets us prove that a Taylor expansion of some function f actually represents f:

1. Assume[2] that the standard formula $d(x^r)/dx = rx^{r-1}$ for the derivative of a power holds for all $r \in \mathbf{R}$ as long as $x > 0$. Use Taylor's theorem with remainder to prove that the binomial expansion

$$(1+x)^r = 1 + rx + r(r-1)\frac{x^2}{2!} + r(r-1)(r-2)\frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \left(\prod_{j=0}^{n-1} (r-j) \cdot \frac{x^n}{n!} \right)$$

holds for all $r \in \mathbf{R}$ and $x \in (-\frac{1}{2}, 1)$, and that the convergence is uniform in compact subsets of $(-\frac{1}{2}, 1)$. [I don't think we get convergence to $(1+x)^r$ for all $x \in (-1,1)$ this way; but see Problem 4.]

Second, while differentiation "doesn't commute" even with uniform convergence of functions, "antidifferetiation" does, in the following sense:

2. Let $f_n : [a,b] \to \mathbf{R}$ be a sequence of $\mathcal{C}^1$ (continuously differentiable) functions whose derivatives f_n' converge uniformly to some function g, uniformly on $[a,b]$ (and thus that g is also continuous). Assume further that $\{f_n\}$ converges uniformly to some $f : [a,b] \to \mathbf{R}$. Prove that this limit function f is differentiable and that $f'(x) = g(x)$ for all $x \in [a,b]$. (Do not use integration here or in the next two problems.)

This lets us prove differentiability of power series without integration, as we illustrate in the next two problems.

3. Deduce that the function $E : \mathbf{R} \to \mathbf{R}$ defined by $E(x) = \lim_{N\to\infty} \sum_{n=0}^{N} x^n/n!$ is differentiable and that $E'(x) = E(x)$ for all x.

[1] Quote taken from Chapter 10 of M. Artin's *Algebra.* It roughly translates as "Please forget all that you have learned in school, for you haven't [really] learned it." Don't complain about the German transcription, which is presumably of some local dialect — even I recognize that this isn't the the German we *auf der Schule lernen.*

[2] This result is true, but it's still a bit tricky for us to prove it. It's not too hard when $r \in \mathbf{Q}$, but the standard method of writing an arbitrary real number as the limit of a sequence of rational numbers is not enough: we could use it to define x^r for any $r \in \mathbf{R}$ but not (without further work) to differentiate it, because in general differentiation does not commute with pointwise or even uniform limits. One route is via the result proved in Problem 2. We shall construct x^r as $\exp(r \log x)$ once we have defined and analyzed the logarithmic and exponential functions, at which point the Chain Rule will let us differentiate it

4. Fix $r \in \mathbf{R}$. For $x \in (-1,1)$ define

$$f(x) = \lim_{N\to\infty} \sum_{n=0}^{N} \left(\prod_{j=0}^{n-1} (r-j) \cdot \frac{x^n}{n!} \right).$$

Show that $f(x)$ is differentiable and $(1+x)f'(x) = rf(x)$. Deduce (again assuming the formula for $d(x^r)/dx$) that $f(x) = (1+x)^r$ for all $x \in (-1,1)$. [Hint: Recall the Wronskian problem from the previous problem set.]

The next few problems are from Rudin pages 138–139:[3]

5. [Rudin #3] Define functions $\beta_j : \mathbf{R} \to \mathbf{R}$ ($i = 1,2,3$) as follows: for each j, set $\beta_j(x) = 0$ for $x < 0$ and $\beta_j(x) = 1$ for $x > 0$; but $\beta_1(0) = 0$, $\beta_2(0) = 1$, $\beta_3(0) = 1/2$. Let $f : [-1,1] \to \mathbf{R}$ be any bounded function.
   a) Prove that $f \in \mathscr{R}(\beta_1)$ <u>iff</u> $f(0) = \lim_{x\to 0+} f(x)$, and then $\int_{-1}^{1} f\, d\beta_1 = f(0)$;
   b) State and prove a similar result for $\mathscr{R}(\beta_2)$;
   c) Prove that $f \in \mathscr{R}(\beta_3)$ <u>iff</u> f is continuous at 0, in which case $\int_{-1}^{1} f\, d\beta_j = f(0)$ for each $j = 1,2,3$.

6. [Rudin #8; "integral test" for convergence of a positive series $\sum_{n>n_0} f(n)$]
   Let $\alpha : [a,\infty) \to \mathbf{R}$ be any increasing function. Suppose $f : [a,\infty)$ is in $\mathscr{R}(\alpha)$ on $[a,b]$ for each $b > a$. The "improper Riemann-Stieltjes integral" $\int_a^\infty f(x)\, d\alpha(x)$ is then defined as $\lim_{b\to\infty} \int_a^b f(x)\, d\alpha(x)$ if the limit exists [and is finite]. In that case we say the integral *converges*; we say it *converges absolutely* if $\int_a^\infty |f(x)|\, d\alpha(x)$ also converges. Naturally the "improper Riemann integral" is the special case of this where $\alpha(x) = x$ for all x. [Likewise for $\int_{-\infty}^a$; and $\int_{-\infty}^\infty f\, d\alpha$ converges to $\int_{-\infty}^0 f\, d\alpha + \int_0^\infty f\, d\alpha$ if both integrals converge.]
   Suppose further that $f(x) \geq 0$ and f is monotone decreasing on $x \geq 1$. Prove that $\int_1^\infty f(x)\, dx$ converges if and only if $\sum_{n=1}^\infty f(n)$ converges.

7. [Integration by parts for improper integrals] Show that in some cases integration by parts can be applied to the "improper" integrals defined in the previous problem; that is, state appropriate hypotheses, formulate a theorem, and prove it. Your hypotheses should be applicable in the following special case: the improper integrals $\int_0^\infty \cos(x)\, dx/(x+1)$ and $\int_0^\infty \sin(x)\, dx/(x+1)^2$ converge and are equal. Show that one of these two integrals (which one?) conerges absolutely, but the other does not.

8. In the `vint` handout on integration of vector-valued functions, you might have expected a theorem to the effect that such a function is integrable (as defined there) with integral I if and only if for each $\epsilon > 0$ there exists a partition P all of whose Riemann sums differ from I by vectors of norm at most ϵ. Certainly the existence of such P is a consequence of integrability, but in fact the converse implication does not hold! Prove this by finding a normed vector space V and a function $f : [0,1] \to V$ such that $\Delta(P) = 1$ for any partition P (and thus $f \notin \mathscr{R}$), but nevertheless for each ϵ there exist partitions P such that every Riemann sum $R(P,\vec{t})$ for $\int_0^1 f(x)\, dx$ has norm at most ϵ. [Hints: f cannot be continuous or even nearly (e.g. piecewise) continuous, because then our vector version of Thm. 6.8 would yield integrability; in fact the function I have in mind is discontinuous everywhere. Moreover, V cannot be finite dimensional. Thus the example is rather pathological — but it is also simple enough that it can be described and proved in a short paragraph.]

This problem set due Friday, 2 March, at the beginning of class.

[3]For the first of these, cf. also Rudin #1: Suppose $\alpha : [a,b] \to \mathbf{R}$ is increasing, and continuous at x_0. Define $f : [a,b] \to \mathbf{R}$ by $f(x) = 0$ if $x \neq x_0$ and $f(x_0) = 1$. Then $f \in \mathscr{R}(\alpha)$ [i.e. f is integrable with respect to α], and $\int_a^b f(x)\, d\alpha = 0$.