Math 55b: Honors Real and Complex Analysis

Homework Assignment #3 (~~9~~ 11 February 2018):
Metrics, sequences, and compactness

Topologist walks into $\overline{A}$. It is closed.
—source unknown[1]

More about the topology of $\mathbf{R}$, and a connection with continuity:

1. i) Prove that the only subsets of $\mathbf{R}$ that are simultaneously open and closed are $\emptyset$ and $\mathbf{R}$.
   ii) Suppose X, Y are metric spaces, and that X has the discrete metric. Find all continuous maps from X to Y. Find all continuous maps from $\mathbf{R}$ to X.

Another characterization of convergence:

2. Fix a sequence $\{r_n\}$ of positive real numbers such that $r_1 > r_2 > r_3 > \cdots$ and $r_n \to 0$. Let $\widetilde{\mathbf{N}}$ be the metric space consisting of $1, 2, 3, \ldots$ together with a symbol ∞, with the distance function defined by
$$d(m,n) = |r_m - r_n|, \quad d(n,\infty) = d(\infty,n) = r_n, \quad d(\infty,\infty) = 0.$$
[In other words, d is defined so that the map $\rho : \widetilde{\mathbf{N}} \to \mathbf{R}$ given by $n \mapsto r_n$, $\infty \mapsto 0$ is an isometry to $\rho(\widetilde{\mathbf{N}})$.] Let $\mathbf{N}$ be the subspace $\{1, 2, 3, \ldots\}$ of $\widetilde{\mathbf{N}}$, so $\widetilde{\mathbf{N}}$ is the disjoint union of $\mathbf{N}$ with $\{\infty\}$.
   i) Which subsets of $\mathbf{N}$ are open? Which subsets of $\widetilde{\mathbf{N}}$ are open?
   ii) Use your answers to (i) to decide whether $\mathbf{N}$ and $\widetilde{\mathbf{N}}$ are compact. (That is: in each case, either exhibit an open cover with no finite subcover, or prove that no such open cover exists.)
   iii) Let $\{s_n\}$ be a sequence in an arbitrary metric space X. Let $\sigma : \mathbf{N} \to X$ be the map that takes n to s_n. Show that $\{s_n\}$ converges if and only if σ extends to a continuous function $\tilde{\sigma} : \widetilde{\mathbf{N}} \to X$ (that is, if and only if there exists a continuous $\tilde{\sigma} : \widetilde{\mathbf{N}} \to X$ such that $\tilde{\sigma}(n) = \sigma(n)$ for all $n \in \mathbf{N}$), in which case $\tilde{\sigma}(\infty) = \lim_{n\to\infty} s_n$.

More about sequences and $\mathcal{C}(X, Y)$:

3. Prove that[2]
$$d_1(f,g) := \int_0^1 |f(x) - g(x)|\, dx$$
is a metric on the space $\mathcal{C}([0,1], \mathbf{C})$ of (bounded) continuous functions $f : [0,1] \to \mathbf{C}$ on the closed unit interval $[0,1]$. [That is, the vector space $\mathcal{C}([0,1], \mathbf{C})$ has a norm $\|\cdot\|_1$ defined by $\|f\|_1 = \int_0^1 |f(x)|\, dx$.]

[1] Apparently this joke was making the rounds at the Joint Math Meetings last month in San Diego (sometimes with the corollary "But possibly also open"). Recall that $\overline{A}$ is pronounced "A bar".

[2] Yes, I know: we have yet to officially define $\int_0^1$. For this problem, though, only the most basic facts are needed, such as the existence of $\int_0^1 F(x)\, dx$ for any continuous function $F : [0,1] \to \mathbf{R}$, and the fact that if $F(x) \le G(x)$ for all $x \in [0,1]$ then $\int_0^1 F(x)\, dx \le \int_0^1 G(x)\, dx$. Only ϵ more is needed for the next problem.

4. Let X be our metric space $\mathcal{C}([0,1],\mathbf{R})$ of continuous functions on $[0,1]$ with $d_X(f,g) = \max_{0\le x\le 1}|f(x)-g(x)|$.
   i) Find an infinite set $S \subset X$ such that the restriction of d_X to S is the discrete metric. Can you do the same for the metric d_1 of the previous problem?
   ii) Let $\mathbf{0} \in X$ be the zero function. Is the closed unit ball $\overline{B}_1(\mathbf{0})$ in X compact? Why?

5. Define sequences $\{f_n\}$, $\{g_n\}$ $(n = 1, 2, 3, \ldots)$ of functions from $\mathbf{R}$ to $\mathbf{R}$ by

$$f_n(x) = \frac{n}{x^2+n^2}, \qquad g_n(x) = \frac{n^2}{x^2+n^2}.$$

   i) Do these sequences of functions f_n and g_n converge pointwise?
   ii) Do they converge uniformly on $\mathbf{R}$? Explain.

More about compact metric spaces:

6. Say that a subset E of a metric space X is "totally bounded relative to X" if, for each $\epsilon > 0$, there is a finite cover of E by ϵ-neighborhoods in X. Prove that E is totally bounded relative to X if and only if E is totally bounded. [That is, allowing centers of the ϵ-neighborhoods to be in a larger ambient metric space does not change the notion of total boundedness. This simplifies the proof of Heine-Borel.]

7. Let $\{U_\alpha\}$ be an open cover of the compact metric space X. Show that there exists $r > 0$ such that, for every $x \in X$, the r-ball $B_r(x)$ is contained in some U_α. [Proceed by contradiction, assuming no such r exists. Construct a sequence $\{x_n\}$ in X such that $B_{1/n}(x_n)$ is contained in no U_α. Let x be the limit of a convergent subsequence. Show that $B_\rho(x) \subseteq U_\alpha$ for some $\rho > 0$ and α. Obtain the contradiction by showing that $B_\rho(x)$ contains some $B_{1/n}(x_n)$.]

8. i) Prove that (as noted but not proved in Math 55a) if V is a finite-dimensional vector space over $\mathbf{R}$ or $\mathbf{C}$ then all norms on V are equivalent (see http://www.math.harvard.edu/~elkies/M55a.17/norm.html if you need a reminder of what this means).
   ii) Use (i) and the result of Problem 3 to solve the following Putnam problem: Prove that for each positive integer n there exists a constant C_n such that, if $P(x)$ is a polynomial of degree at most n, then $|P(0)| \le C_n \int_{-1}^{1} |P(x)|\,dx$.

9. Prove that the closed unit ball in an inner-product space V over either $\mathbf{R}$ or $\mathbf{C}$ is compact (in the metric topology associated to the usual norm defined by $\|x\| = (x,x)^{1/2}$) if and only if V is finite dimensional.

This problem set is due Friday, February 16, at the beginning of class. But since it came out two days late, you may (without penalty) postpone any one or two of these problems until the due date of the next problem set.