Math 55b: Honors Real and Complex Analysis

Homework Assignment #11 (16 April 2018):
Complex analysis, cont'd

> ... because it leaves a residue at every pole.
> *—punchline on an old math joke that has nothing to do with the complete graph with 36 edges*

> [Legendre introduced the notation $\Gamma(s)$ for Euler's $\Pi(s-1)$ so that he] could do with the gamma function what the Catholic church did 170 years later: He put a simple pole at the origin.
> *—"answer" of Greg Kuperberg '87 to Kevin Casto's Mathoverflow Question 20960 "Why is the Gamma function shifted from the factorial by 1?" (October 2010). Pope John Paul II was a Pole; I don't know why Greg would deem him "simple" except for the sake of this joke.*

One further standard application, and one not-so-standard application, of contour integration to the evaluation of definite integrals:

1. [From our graduate Qualifying Exam, Spring 1997] For $b > 0$, compute $\int_0^\infty \log x \; dx/(x^2 + b^2)$. [Use contour integration. There is a "Putnam trick" to solve this with just single-variable real calculus, but that by itself should not get full credit — though you might still use it to check your answer.]

2. Recall that for $z \in \mathbf{C}$ the *hyperbolic cosine* $\cosh z$ is defined as $\cos(iz) = (e^z + e^{-z})/2$. [1] Prove that
$$\int_0^\infty \frac{\cos(mx)}{\cosh(\pi x)} \, e^{-mx^2} dx = \frac{1}{2} \, e^{-m/4}$$
for all $m \geq 0$. [From the Fall 1998 Qualifying Exam.] Can you evaluate any other such integrals this way (other than those obtained trivially from this formula by linear change of variable etc.)?

More about contour integrals, residues, and the complex Gamma function:

3. Prove that the entire function $f(s) = 1 - 2^{-s} - 3^{-s} - 6^{-s}$ has one simple zero in each horizontal strip $(\log 6/(2\pi)) \operatorname{Im}(s) \in (n - \frac{1}{2}, n + \frac{1}{2})$ $(n \in \mathbf{Z})$, and these zeros all satisfy $|\operatorname{Re}(s)| \leq 1$ with equality only for the zero at $s = 1$. Can you prove that for each $\epsilon > 0$ there are infinitely many zeros of real part $> 1 - \epsilon$?

4. Determine for any entire function f the residue at $z = 0$ of the differential $f(\cot(z))\, dz$. In particular, what is the residue at the origin of $\sin(\cot(z))\, dz$? [NB in general $f(\cot(z))$ has an essential singularity at $z = 0$. As far as I know, the other odd-order coefficients of the Laurent expansion of $\sin(\cot(z))$ about $z = 0$ are not known in closed form.]

5. Use contour integration to find, for any $c \in \mathbf{C}$, the coefficients a_n of the analytic function $w(z) = \sum_{n=1}^\infty a_n z^n$ on a neighborhood of $z = 0$ such that $w(z)(1 - z)^c = z$. [The expression $(1 - z)^c$ can be unambiguously defined near $z = 0$ by
$$(1 - z)^c = \exp(c \log(1 - z)) = \exp(-c \sum_{m=1}^\infty z^m/m).$$
Check that your answer agrees with the elementary formulas for $c = 0, \pm 1, -2, -1/2$.]

[1] I think I mentioned that the abbreviation "cosh" is from the Latin <u>*cos*</u>*inus* <u>*h*</u>*yperbolicus*, but is still often pronounced to rhyme with "mosh" or even expanded to "coshine"... This function, and "sinh" (hyperbolic sine, sometimes pronounced "sinch" — rhyming with "inch" rather than "synch"), are called "hyperbolic" because $\{(\cosh t, \sinh t) : t \in \mathbf{R}\}$ is a branch of the hyperbola $x^2 - y^2 = 1$, as the "circular trigonometric functions" cos and sin give the circle $x^2 + y^2 = 1$. There are thus also $\tanh = \sinh / \cosh$ ("tanch") and $\operatorname{sech} = 1/\cosh$ (I have yet to hear "setch").

6. Prove that for $c, x > 0$ the integral $\int_{-\infty}^{\infty} \Gamma(c+iy)\, x^{-iy}\, dy$ converges to $2\pi x^c e^{-x}$. [While we shall not prove Stirling's approximation for the complex Gamma function, here it will be enough to use judiciously the elementary inequality $|\Gamma(z)| \leq |\Gamma(\mathrm{Re}(z))|$ for $\mathrm{Re}(z) > 0$.]

From the handout on the Hadamard product formula:

7.–9. Solve Exercises 3, 4, and 7.

The next problems concern functions to and from the Riemann sphere, which can be defined as the "projective line" $\mathbf{P}^1(\mathbf{C}) = (\mathbf{C}^2 - \{(0,0)\})/\mathbf{C}^*$. An analytic map f from some open $E \subset \mathbf{C}$ to $\mathbf{P}^1(\mathbf{C})$ is a pair (f_1, f_2) of analytic functions on E that do not have a common zero in E; another such pair (g_1, g_2) gives the same function if the vectors $(f_1(z), f_2(z))$ and $(g_1(z), g_2(z))$ are proportional for all $z \in E$. You should check that every z has a neighborhood in which f has a representative such that either f_1 or f_2 is the constant function 1. We usually identify f with f_1/f_2, with the understanding that this may take the value ∞ (when $f_2 = 0$). The idea is that $\mathbf{P}^1(\mathbf{C})$ is made up of two copies of $\mathbf{C}$, one represented by vectors $(z_1, 1)$ and the other by $(1, z_2)$, and intersecting on $\mathbf{C}^*$ with $z_2 = z_1^{-1}$. Since the map $z_1 \mapsto z_1^{-1}$ is analytic on $\mathbf{C}^*$ it makes sense to say that an analytic or meromorphic map F *from* $\mathbf{P}^1(\mathbf{C})$ to $\mathbf{C}$ or $\mathbf{P}^1(\mathbf{C})$ is one whose restriction to each of these two copies of $\mathbf{C}$ is analytic or meromorphic respectively.[2]

10. Show that an analytic map $E \to \mathbf{P}^1(\mathbf{C})$ other than the constant map ∞ is a meromorphic function on E; and that conversely if $E = \mathbf{C}$ then any meromorphic function on E is an analytic map $E \to \mathbf{P}^1(\mathbf{C})$. (This last is true for any open subset of $\mathbf{C}$ but we cannot prove it in this generality.) Show moreover that the meromorphic functions on $\mathbf{P}^1(\mathbf{C})$ are precisely the rational functions (note that the rational functions of z_1 are the same as the rational functions of $z_2 = z_1^{-1}$ so this is well-defined). [Note that our generalization of Liouville's theorem to entire functions bounded by $C|z|^n$ as $|z| \to \infty$ is the special case of a meromorphic function with no poles except possibly at ∞.]

11. Let $A, B \in \mathbf{C}[z]$ be polynomials such that B has distinct roots $z_1, \ldots, z_n$. Let ω be the differential $(A(z)/B(z))\, dz$ on $\mathbf{C} - \{z_1, \ldots, z_n\}$. Show that the residue of ω at each z_j is $A(z_j)/B'(z_j)$. Conclude that if $\deg(A) \leq \deg(B) - 2$ then $\sum_{j=1}^{n} A(z_j)/B'(z_j) = 0$. What happens if $\deg(A) = \deg(B) - 1$?

Since these identities are purely algebraic results, they must hold for polynomials over any algebraically closed field; but — as with invariance of the residue under coordinate change — a direct algebraic proof, though possible, is harder and less revealing.

12. Let E be the open right half-plane $\{z \in \mathbf{C} : \mathrm{Re}(z) > 0\}$, and $f : E \to \mathbf{C}$ a bounded analytic function. Suppose $f(x_k) = 0$ for some real numbers x_k with $0 < x_1 \leq x_2 \leq x_3 \leq \cdots$. Prove that
$$|f(1)| \leq \left(\prod_{k=1}^{n} \frac{|1 - x_k|}{1 + x_k} \right) B$$
where $B = \sup_{z \in E} |f(z)|$, and find all f for which equality is attained. Deduce that if f is not identically zero but vanishes at $x_k > 0$ for each $k = 1, 2, 3, \ldots$ then $\sum_{k=1}^{\infty} 1/x_k < \infty$.

This underlies one approach to the proof of the Müntz-Szász theorem: if the topological span of $\{t^{x_k}\}_{k=1}^{\infty}$ is not dense in $\mathcal{C}(0,1)$ then its L^2 closure is a proper subspace of $L^2(0,1)$, and then there's a nonzero $\phi \in L^2(0,1)$ orthogonal to each t^{x_k}; then $f(x) := \int_0^1 \phi(t)\, t^x\, dt$ is holomorphic and bounded on E and vanishes at $x = x_k$ for each k, "etc.". We didn't officially define $L^2(0,1)$ in this year's Math 55: it's the completion of the inner product space $\mathcal{C}([0,1])$ with respect to its norm $\|f\| = \left(\int_0^1 f(x)^2\, dx\right)^{1/2}$.

This problem set due Monday, April 30 at 5PM.

[2] This is basically saying that $\mathbf{P}^1(\mathbf{C})$ is a 1-dimensional complex *manifold*.