Math 55b: Honors Real and Complex Analysis

Homework Assignment #10 (9 April 2018):
Integration in $\mathbf{R}^k$; more analysis in $\mathbf{C}$

Old MacDonald had a form: $e_i \wedge e_i = 0$.
—Mike Stay (October 2009), in a `mathoverflow` *thread "Do good math jokes exist?".*

Change of variable and related ideas:

1. Let $Q : \mathbf{R}^n \to \mathbf{R}$ be a positive-definite form. Show that the integral of $e^{-Q(x)}$ over $x \in \mathbf{R}^n$ converges, and evaluate this integral. For $y \in \mathbf{R}^n$ determine the integral of $\exp(-Q(x)+i\langle x,y\rangle)$ over $x \in \mathbf{R}^n$, where $\langle x,y\rangle = \sum_{i=1}^n x_i y_i$ is the usual inner product.

2. Let n be a positive integer, V the vector space of symmetric $n \times n$ matrices, and $E \subset V$ the set of positive-definite matrices. This set is convex: if A, B are positive-definite then so is $(A+B)/2$. Prove that $\det((A+B)/2)^2 \geq (\det A)(\det B)$ for all $A, B \in E$, with equality <u>iff</u> $A = B$. [Use the first part of Problem 1. I learned this proof from Don Zagier; I do not know its original source. This determinant inequality is equivalent to the statement that the function $A \mapsto 1/\det(A)$ on E is "logarithmically convex".]

3. [Calabi] Prove that
$$\int_0^1\!\!\int_0^1 \frac{dx\,dy}{1-x^2y^2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{3}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}.$$
(Note that this is an improper integral (why?) so some care will be needed here.) Now let Δ be the triangle $\{(u,v) \in \mathbf{R}^2 | u, v > 0, u+v < \pi/2\}$. Prove that the map $T : \Delta \to \mathbf{R}^2$ defined by $T(u,v) = (\sin u/\cos v, \sin v/\cos u)$ is a $\mathscr{C}^1$ and $\mathscr{C}^1$-invertible map of Δ to the open unit square $0 < x, y < 1$, and compute its Jacobian determinant. Conclude that
$$\int_0^1\!\!\int_0^1 \frac{dx\,dy}{1-x^2y^2} = \iint_\Delta 1\,du\,dv = \frac{\pi^2}{8}$$
as desired.
[One way to prove $T(\Delta)$ covers the square is to construct for each (x,y) a suitable contraction map on the square $[0,\pi/2]^2$. Can you generalize this to evaluate $\sum_{n=1}^\infty 1/n^4$, or show more generally that $\sum_{n=1}^\infty 1/n^s$ is a rational multiple of π^s for all positive even s? Can you prove that
$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + - \cdots = \frac{\pi^3}{32}$$
using this method?]

4. [Newton] Let V be the vector space $\mathbf{R}^3$ with the usual Euclidean norm, and let $\omega = dx_1 \wedge dx_2 \wedge dx_3$ be the standard volume form on V. Define a vector-valued function g from $V - \{\mathbf{0}\}$ to V by $g(v) = v/|v|^3$. Fix $r_1, r_2 \in \mathbf{R}$ with $0 < r_1 < r_2$, and let E be the spherical shell $\{x \in V : r_1 < |x| < r_2\}$.
   i) Prove that if $v_0 \in V$ with $|v_0| < r_1$ then $\int_E g(x - v_0)\,\omega = 0$.
   ii) Prove that if $v_0 \in V$ with $|v_0| > r_2$ then $\int_E g(x-v_0)\,\omega = Ig(-v_0)$, where $I = \int_E \omega = 4\pi(r_2^3 - r_1^3)/3$ is the volume of E.

Note: The manipulations required, while straightforward, may be somewhat lengthy. I don't assign many such problems, but this one has special significance, both theoretical and historical: Newton had observed that the force of gravity near the earth's surface and the acceleration of the moon in its orbit around the earth are consistent with a universal inverse-square law of gravitation, provided the gravitational force of a spherical body was equivalent to that of an equal point mass; but it was only some twenty years later that he succeeded in proving this result and thus clinching the inverse-square law. That's the significance of part (ii); part (i) also figures in the physics of electrostatic forces: a uniformly charged sphere exerts no force on its interior. (For the effect on the exterior of the sphere, see again part (ii).) It is now known that this can be proved in a more "conceptual" way, albeit at the cost of introducing more machinery (surface integrals, etc.), from the fact that the inverse-square force exerted by a point mass is the gradient of a potential function $G(x) = C/|x - x_0|$ satisfying the Laplace equation $\Delta G(x) = 0$ for all $x \neq x_0$.

A bit more about the structure of analytic functions, now that we know that for functions of a complex variable "analytic" is equivalent to "differentiable":

5. (Singularities of analytic functions)
   Suppose f is analytic on a "punctured disc" $B_r^*(z_0) := \{z \in \mathbf{C} : 0 < |z - z_0| < r\}$.
   i) If f is bounded on $B_r^*(z_0)$ then it extends uniquely to an analytic function on $B_r(z_0)$; that is, there is a unique analytic function $\tilde{f} : B_r(z_0) \to \mathbf{C}$ such that $\tilde{f}(z) = f(z)$ for all $z \in B_r^*(z_0)$. [Define $g : B_r(z_0) \to \mathbf{C}$ by $g(z) = (z - z_0)^2 f(z)$ for $z \in B_r^*(z_0)$, and $g(z_0) = 0$; show that g is differentiable and thus analytic, etc. Such z_0 is thus called a "removable singularity" of f.]
   ii) If f is unbounded, but there is some integer N such that $(z - z_0)^N f(z)$ is bounded, show that $f(z)$ has a power-series expansion $\sum_{n=-N}^{\infty} a_n (z - z_0)^n$ convergent on $B_r^*(z_0)$. [Such z_0 is called a "pole" of f, specifically a "pole of order N_0" where N_0 is smallest possible N, which is also $\max\{m : a_{-m} \neq 0\}$.]
   iii) If there is no such N, show that the image of $B_s^*(z_0)$ is dense in $\mathbf{C}$ for all positive $s < r$. [If not, the image omits some open ball; now transform f so you can apply (i). Such z_0 is called an "essential singularity" of f; a paradigmatic example is $f(z) = e^{1/z}$ with $z_0 = 0$.]

6. (Reflection principles)
   i) Let E be an open set in $\mathbf{C}$, and $E' = \{z \in \mathbf{C} : \bar{z} \in E\}$ (I can't call this "$\overline{E}$" because that looks like topological closure). Prove that $f : E \to \mathbf{C}$ is differentiable if and only if the function $E' \to \mathbf{C}$ defined by $z \mapsto \overline{f(\bar{z})}$ is differentiable. Deduce that if E is an open rectangle or circle symmetric about the real axis and $f(z) \in \mathbf{R}$ for all $z \in E \cap \mathbf{R}$ then $f(\bar{z}) = \overline{f(z)}$ for all $z \in E$.
   ii) Suppose now that $r > 1$ and let E be the annulus $\{z \in \mathbf{C} : 1/r < |z| < r\}$. If $f : E \to \mathbf{C}$ is a differentiable function such that $|f(z)| = 1$ for all z on the unit circle $|z| = 1$, what can you deduce about f?
   [Hint: an analytic function on $B_r(z_0)$ is either identically zero or of the form $(z - z_0)^n f(z)$ for some integer $n \geq 0$ and some analytic f such that $f(z_0) \neq 0$.]

7. Let E be an open set in $\mathbf{C}$, fix some real number B, and let $\mathcal{F}$ be the family of analytic functions $f : E \to \mathbf{C}$ such that $|f(z)| \leq B$ for all $z \in E$. If K is any compact subset of E, prove that $\{f'(z) : f \in \mathcal{F},\ z \in K\}$ is bounded. Deduce that if E is convex then "$\mathcal{F}|_K$ is equicontinuous": for all $\epsilon > 0$ there exists $\delta > 0$ such that $|f(z) - f(z')| < \epsilon$ for all $f \in \mathcal{F}$ and any $z, z' \in K$ such that $|z - z'| < \delta$.

This problem set due Monday, April 16, at the beginning of class.