Math 55b: Honors Advanced Calculus and Linear Algebra

Homework Assignments #1 and #2 (January 2018):
Metric topology basics

"I'm sorry…"

"Don't topologize."

—Martin Gardner (adapted)

Definition and constructions of metric spaces:

1. [Cf. Rudin, p.44, Ex.10] For any set X define the *discrete metric* on X by $d(p, q) = 0$ if $p = q$ and $d(p, q) = 1$ if $p \neq q$. Prove that this is indeed a metric. With this metric, which subsets of X are open? Which are closed? Which are dense?

2. Let (X, d) be a metric space. Define $d_0(x, y) := d(x, y)/(1 + d(x, y))$ for all $x, y \in X$.
   i) Prove that d_0 is also a metric on X.
   ii) Prove that a subset of X is open under the metric d if and only if it is open under d_0. [Thus (X, d) and (X, d_0) are the same as "topological spaces", but generally not isometric (identical as metric spaces); see Problem 7 below.]
   iii) Show that the metric space (X, d_0) is always bounded, even though (X, d) might not be.

3. [Cf. Rudin, p.44, Ex.11] Which of the following defines a metric on $\mathbf{R}$? Explain.
   i) $d_1(x, y) := (x - y)^2$
   ii) $d_2(x, y) := \sqrt{|x - y|}$
   iii) $d_3(x, y) := |x^2 - y^2|$
   iv) $d_4(x, y) := |x^3 - y^3|$
   v) $d_5(x, y) := |x - 2y|$
   vi) $d_6(x, y) := |x - y|/(1 + |x - y|)$

4. Suppose X is a set and $d : X \times X \to \mathbf{R}$ is a function satisfying all the distance axioms except that $d(p, q) = 0$ need not imply $p = q$.
   i) Check that the following is an example of such a pair (X, d): let $X = \mathbf{R}^3$ and
   $$d\big((x_1, x_2, x_3), (x_1', x_2', x_3')\big) := \max(|x_1 - x_1'|, |x_2 - x_2'|).$$
   NB You should solve parts (ii)–(iv) for any such (X, d), not just this example with $X = \mathbf{R}^3$.
   ii) For $p, q \in X$ define $p \sim q$ to mean $d(p, q) = 0$. Prove that this is an *equivalence relation*: $p \sim p$ for all $p \in X$, $p \sim q \Rightarrow q \sim p$, and $p \sim q$, $q \sim r \Rightarrow p \sim r$ [Rudin, Definition 2.3, p.25].
   iii) Show that if $p \sim p'$ and $q \sim q'$ then $d(p, q) = d(p', q')$.
   iv) Let $\widetilde{X}$ be the set of *equivalence classes*, i.e., subsets of X of the form $[p]$, defined as $[p] := \{p' \in X : p \sim p'\}$. [NB $[p] = [p'] \iff p \sim p'$.] Part (iii) showed that
   $$\tilde{d}([p], [q]) = d(p, q)$$
   is a *well-defined function* $\tilde{d} : \widetilde{X} \times \widetilde{X} \to \mathbf{R}$; that is, for all $P, Q \in \widetilde{X}$ the value of $\tilde{d}(P, Q)$ does not depend on the choice of representatives of the equivalence classes P, Q. Prove that $\tilde{d}(\cdot, \cdot)$ satisfies the axioms of a metric.
   v) Part (iv) makes $\widetilde{X}$ a metric space. What is this metric space for our above example with $X = \mathbf{R}^3$?

Problem 4 may/should remind you of Problem 3 on PS7 of Math 55a.

Problems 5–7 concern isometries betwen metric spaces. Recall that an *isometry* between metric spaces X, Y is a bijection $i : X \to Y$ such that

$$d_Y(i(x_1), i(x_2)) = d_X(x_1, x_2)$$

for all $x_1, x_2 \in X$. If $X = Y$ we say i is an "isometry of X" .

5. i) Prove that: the identity map on a metric space is an isometry; if $i : X \to Y$ is an isometry, then so is the inverse map $i^{-1} : Y \to X$; and if $i : X \to Y$ and $j : Y \to Z$ are isometries, then so is the composite map $j \circ i : X \to Z$. Deduce that the isometries of a metric space constitute a group under composition of functions.
   ii) For $X = Y = \mathbf{R}$, the function $i(x) = -x$ is an isometry, as is $j_a(x) = x + a$ for any $a \in \mathbf{R}$.
   iii) Every isometry from $\mathbf{R}$ to itself is either j_a or $i \circ j_a$ for some a.

6. i) Let V be a finite-dimensional real inner product space. Prove that the isometries of V are the maps $x \mapsto A(x + u)$ with $u \in V$ and A in the orthogonal group $\mathrm{O}(V)$.
   ii) Determine the group of isometries of $\mathbf{R}^n$ equipped with the sup metric.

7. Let (X, d) be a metric space, and (X, d_0) the bounded metric space of Problem 2 [with the same X, and $d_0 = d/(1 + d)$].
   i) Prove that (X, d_0) is isometric with (X, d) if and only if X has at most one element. (Warning: this means you must prove that *no* map from X to itself is an isometry, not just that the identity map is not an isometry!)
   ii) Construct an example of an infinite metric space (X, d) and a map $i : X \to X$ satisfying

$$d(i(x_1), i(x_2)) = d_0(x_1, x_2)$$

   for all $x_1, x_2 \in X$. [That is, i is an isometry between (X, d_0) and $(i(X), d|_{i(X)})$.]

Closures, etc.:

8. [Rudin, p.43, Ex.6] Let E be a subset of a metric space, and E' its set of limit points. Prove that E' is closed, and that E and $\bar{E}$ have the same limit points. (Recall that $\bar{E}$, the *closure* of E, is defined by $\bar{E} = E \cup E'$.) Is it true that E and E' have the same limit points for every E?

9. [Rudin, p.43–4, Ex.5,13]
   i) Construct a bounded closed subset of $\mathbf{R}$ with exactly three limit points.
   ii) [This is rather trickier] Construct a bounded closed set $E \subset \mathbf{R}$ for which E' is an infinite countable set.

10. [Rudin, p.43, Ex.7] Let $A_1, A_2, A_3, \ldots$ be subsets of a metric space.
   i) If $B_n = \cup_{i=1}^n A_i$, prove that $\bar{B}_n = \cup_{i=1}^n \bar{A}_i$ (the closure of a finite union is the union of the closures).
   ii) If $B = \cup_{i=1}^\infty A_i$, prove that $\bar{B} \supseteq \cup_{i=1}^\infty \bar{A}_i$ (the closure of a countable union contains the union of the closures).
   iii) Give an example where this inclusion is proper (a.k.a. strict), that is, an example of a metric space with subsets A_i such that $\bar{B} \neq \cup_{i=1}^\infty \bar{A}_i$.

Two different notions of distance between subsets of a metric space:

11. [Distance between subsets of a metric space] For any two nonempty subsets A, B of a metric space X, define the *distance* $d(A, B)$ between A and B by

$$d(A, B) := \inf\{d(x, y) : x \in A,\ y \in B\}.$$

Prove that for any subsets A, B, C of X and any element $x \in X$ we have:
i) $d(\bar{A}, \bar{B}) = d(A, B)$ (where $\bar{A}, \bar{B}$ are the closures of A, B respectively);
ii) $d(\{x\}, A) = 0$ if and only if $x \in \bar{A}$;
iii) $d(A, B \cup C) = \min\{d(A, B), d(A, C)\}$;
iv) $d(A, \{x\}) + d(\{x\}, B) \geq d(A, B)$.
Is it true that the triangle inequality $d(A, C) + d(C, B) \geq d(A, B)$ holds for all A, B, C?

12. [Minkowski distance between nonempty bounded closed subsets of a metric space] Recall that $N_r(x)$ is the radius-r neighborhood of x, a.k.a. the open ball of radius r about x. For a subset A of a metric space X, and a positive real number r, define

$$N_r(A) := \bigcup_{x \in A} N_r(x).$$

One may visualize $N_r(A)$ as the radius-r neighborhood of A. For instance, $N_r(\emptyset) = \emptyset$; $N_r(\{x\}) = N_r(x)$; $N_r(X) = X$; and $r' \geq r \Rightarrow N_{r'}(A) \supseteq N_r(A)$.

For two *nonempty, bounded, closed* subsets A, B of a metric space X, define the *Minkowski distance* $\delta(A, B)$ between A and B by

$$\delta(A, B) := \inf\{r : N_r(A) \supseteq B \text{ and } N_r(B) \supseteq A\}.$$

Prove that this defines a metric on the space of nonempty, bounded, closed subsets of X. (You may have noticed that the triangle inequality holds even without the requirement that our bounded nonempty subsets be closed. Why then must our metric space consist only of the closed subsets?)