**Math 55b: Honors Advanced Calculus and Linear Algebra**

Introduction to Hilbert Space
I: Definition, examples, and orthonormal topological bases

To place Fourier analysis in its proper context, we must introduce the notion of a *Hilbert space*, which is also the setting for quantum mechanics and plays an important role in many other branches of mathematics. Hilbert spaces are for most purposes the most natural generalization of finite-dimensional geometry to vector spaces which need not have a finite basis. The definition is:

**Definition.** *A* Hilbert space *is a complete inner product space.*

Of course "complete" means "complete with respect to the distance function $d(x,y) = \sqrt{(x-y,\, x-y)}$ coming from the inner product structure".

The field $\mathbf{F}$ of scalars may be either $\mathbf{R}$ or $\mathbf{C}$; if we must specify $\mathbf{F}$ we speak of a real or complex Hilbert space, respectively. Most of the basic results apply equally in either case; for Fourier analysis, complex scalars are essential.

**Examples.** Any finite-dimensional inner product space is a Hilbert space. The canonical example of an infinite-dimensional Hilbert space is $l_2$. This is the vector space of sequences $\{a_n\}_{n=1}^{\infty}$ with $a_n \in \mathbf{F}$ and $\sum_{n=1}^{\infty} |a_n|^2 < \infty$. The inner product is defined by $(a,b) = \sum_{n=1}^{\infty} a_n \overline{b_n}$.

Two examples of infinite-dimensional inner product spaces which are not complete, and thus not Hilbert spaces, are the subspace $l_2^{(0)} \subset l_2$ consisting of sequences with finitely many nonzero terms, and the space $\mathcal{C}([0,1])$ with the inner product $(f,g) = \int_0^1 f(x)\overline{g(x)}\,dx$. However, starting from any inner-product space $V$ we may complete it to obtain a Hilbert space. More precisely:

**Proposition.** *Let $V$ be any inner product space, and $\mathcal{H}$ its completion with respect to the distance function $d(x,y) = (x-y, x-y)^{1/2}$. Then the vector space operations and inner product on $V$ extend uniquely to continuous functions on $\mathcal{H}$, $\mathcal{H} \times \mathbf{F}$, or $\mathcal{H} \times \mathcal{H}$ as appropriate, giving $\mathcal{H}$ the structure of a Hilbert space.*

This is easy since the vector space operations and inner product on $V$ are continuous, so extend uniquely to $\mathcal{H}$, and satisfy the same identities by continuity. We have seen already that the completion of any metric space is complete. To verify that $\mathcal{H}$ is a Hilbert space, we need only check that the inner product remains positive definite. But if $v \in \mathcal{H}$ is a vector with $(v,v) = 0$, we can write $v = \lim_{n\to\infty} v_n$ with $v_n \in V$. Then $0 = (v,v) = \lim_{n\to\infty}(v_n, v_n) = \lim_{n\to\infty}(d(v_n,0))^2$, whence $v = 0$ as desired. $\Box$

For instance, the completion of $l_2^{(0)}$ is $l_2$. The completion of $\mathcal{C}([0,1])$ is known as $L_2([0,1])$.[1] We shall soon see that as a Hilbert space each $L_2([a,b])$ is isomorphic

[1] An element of $L_2([0,1])$ is often called an "$L_2$ function", even though it is really an equiv-

with $l_2$; the entire theory of Fourier series can be viewed as the investigation in depth of a particular identification between these two spaces.

**Orthonormal topological bases (ontb's).** Let $V, W$ be any normed vector spaces. We saw last term that a linear map $T : V \to W$ is determined by the images of a spanning set of $V$. For normed vector spaces, we're usually interested not in arbitrary linear maps but in *continuous* ones, and such a map is determined by its images on a *topologically spanning set.* In general, the *topological span* $\overline{sp}(S)$ of a subset $S \subset V$ is the closure of the linear span $sp(S)$ of $S$; this is a vector subspace of $V$, and is complete if $V$ is. We say that $S$ topologically spans $V$ if $V = \overline{sp}(S)$.

If $V$ is finite-dimensional, we have seen that $S$ always contains a minimal spanning set, which we called a "basis". We would like to define "topological basis" in the same way, but matters are not so simple when $V$ is infinite-dimensional. For instance, it easily follows from the Weierstrass approximation theorem that $\{1, x, x^2, \ldots\}$ topologically spans $L_2([0, 1])$; but we shall see that no minimal spanning set can be extracted — indeed in *any* spanning subset *each* element is in the topological span of its complement!

But the notion of an *orthogonal* basis does extend nicely:

**Definition.** *An* orthogonal topological basis *for an inner-product space $V$ is an orthogonal subset of $V$ which topologically spans $V$. An* orthonormal topological basis (ontb) *for $V$ is an orthogonal basis consisting of unit vectors.*

That is, an orthogonal set $S$ is a topological basis if linear combinations of $S$ come arbitrarily close to any vector in $V$. We shall sometimes say "algebraic basis" (or algebraic span, etc.) when we wish to emphasize that we are considering only the (finite) linear combinations and not their closure.

**Examples.** If $S$ is an orthogonal topological basis then $\{s/|s| : s \in S\}$ is an ontb. If $V$ is finite dimensional, an orthogonal (orthonormal) topological basis is the same as an orthogonal (orthonormal) basis. The unit vectors constitute a both an orthonormal basis and an ontb for $l_2^{(0)}$; they constitute an ontb, but not an algebraic basis, for $l_2$, for instance because the $l_2$ vector $(1, 1/2, 1/3, 1/4, \ldots)$ is not a finite linear combination. In general, a Hilbert space can never be algebraically generated by an infinite orthogonal set, for much the same reason. In $L_2([0, 1])$, we have already observed that $\{e^{2\pi in} : n \in \mathbf{Z}\}$ is an orthogonal set; our first step in developing Fourier series will be to show that it is in fact an ontb.

For convenience we state the following basic result only for ontb's; but as usual we can get it for a general orthogonal topological basis $S$ via the associated ontb $\{s/|s| : s \in S\}$.

alence class of sequences of functions and in general there may not be a single function that represents it, or there may be many equivalent ones and no natural choice of representative. . .

**Theorem.** *Let $S$ be an ontb for an inner-product space $V$. For any $v \in V$ there are at most countably many $s \in S$ such that $(v, s) \neq 0$. The sum over such $s$ of $|(v, s)|^2$ converges to $|v|^2$, and the sum of $(v, s)s$ converges to $v$, in the sense that if the sum is infinite then the partial sums of any rearrangement converge to $v$. If $V$ is complete and $s_n$ are distinct elements of $S$ then a sum $\sum_n c_n s_n$ converges if and only if $\sum_n |c_n|^2 < \infty$, in which case any rearrangement converges to the same sum. In that case $V$ consists of all such sums, two of which coincide if and only the coefficient of each $s \in S$ is the same.*

The formulas for $v$ and $|v|^2$ are known respectively as the Fourier expansion of $v$ (relative to $S$) and the Parseval identity — though Fourier and Parseval originally obtained them only for Fourier series.

*Proof*: The key to all this is the known fact that for any finite subset $S' \subset S$, the element of $sp(S')$ nearest to $v$ is the orthogonal projection $\sum_{s \in S'} (v, s)s$, whose norm $\sum_{s \in S'} |(v, s)|^2$ is $\leq |v|^2$. It follows in particular that for each $m = 1, 2, 3, \ldots$, there can be at most $|v|^2 m^2$ elements $s \in S$ for which $|(v, s)| \geq 1/m$ — and thus at most countably many for which $(v, s) \neq 0$. The convergence of $\sum_{s \in S} |(v, s)|^2$ also follows. Now since $S$ is an ontb, $v$ is the limit of finite linear combinations of $S$; if we replace each such combination $\sum_{s \in S'} c_s s$ by $\sum_{s \in S'} (v, s)s$, which is at least as close to $v$, the resulting sequence must still converge; enlarging $S'$ in the latter sum yields a sum at least as close to $v$; therefore $\sum_{s \in S} (v, s)s$ converges to $v$ however rearranged, and comparing norms yields $\sum_{s \in S} |(v, s)|^2$. For any $c_n, s_n$, the partial sums of $\sum_n c_n s_n$ constitute a Cauchy sequence if and only if $\sum_n |c_n|^2 < \infty$. If $V$ is complete, then, the partial sums converge to a vector $v$ such that $(v, s_n) = c_n$ and $v$ is orthogonal to any $s \in S$ that is not one of the $s_n$. The rest of the theorem then follows by applying our results thus far to $v$. For instance, $v = 0$ if and only if $0 = |v|^2 = \sum_n |c_n|^2$ if and only if each $c_n$ vanishes. $\square$

In particular, any countably infinite ontb for a Hilbert space identifies the space isometrically with $l_2$. In the next handout we obtain an intrinsic topological characterization of $l_2$, and as a consequence show that every ontb for $l_2$ is countable. In general all ontb's of a given Hilbert space have the same cardinality, but this is harder to show once the cardinality exceeds $\aleph_0$.