Math 55b: Honors Advanced Calculus and Linear Algebra

Homework Assignment #10 (11 April 2003):
Fourier II

> The correct transliteration of Tchebychev's Russian name is a matter of some controversy. Phillip Davis has written a charming book in which this forms the central theme. Without accepting Davis's preferred spelling I do agree with him that only admirers of Čaykovskiy's music are entitled to write Čebysev.[1]

A few problems on the computation of simple Fourier series, including yet another evaluation of $\zeta(2)$ etc.

1. Determine the Fourier series of the function $f : \mathbf{T} \to \mathbf{R}$ given on $|x| \leq \pi$ by $f(x) = e^{c|x|}$ (with c a real constant). Use this to evaluate in closed form the sum $\sum_{n=1}^{\infty} 1/(n^2 + a^2)$ for $a \in \mathbf{R}$. Check that your answer agrees with the numerical value $\sum_{n=1}^{\infty} 1/(9n^2 + 1) = .171\ldots$

2. i) Determine the Fourier series of the function $f : \mathbf{T} \to \mathbf{R}$ given on $[0, 2\pi]$ by $f(x) = x(2\pi - x)$.
   ii) For each integer $n > 1$, the Fourier series whose e^{irt} coefficient is r^{-n}, except $r = 0$ when the coefficient is 0, converges to a continuous function P_n on $\mathbf{T}$ (by Thm. 9.2 in Körner). Prove that, considered as a function on $[0, 2\pi]$, this P_n is a polynomial of degree n.

3. i) Describe these P_n in terms of the polynomials B_m introduced in the problem 5 of the second 55b problem set.
   ii) Show that the sums $\sum_{r=1}^{\infty} r^{-n}$ (for n even) and $\sum_{r=0}^{\infty}(-1)^r(2r+1)^{-n}$ (for n odd) can be computed by evaluting P_n at particular values of t. Deduce that these sums are rational multiples of π^n.

If P_n are orthogonal polynomials for $(f, g) = \int_a^b f\bar{g}(x)\, d\alpha(x)$, and $\sum_{n=0}^{\infty} a_n P_n$ is the orthogonal expansion of some function f [so $a_n = (f, P_n)/(P_n, P_n)$], then the m-th partial sum is $\int_a^b f(y) K_m(x, y)\, d\alpha(y)\, dy$ where

$$K_m(x, y) = \sum_{n=0}^{m} \frac{P_n(x) P_n(y)}{(P_n, P_n)}.$$

[1] Körner, *Fourier Analysis*, p.200 (conclusion of Chapter 42: "Linkages"). There seems to be a missing haček in this transliteration; possibly Davis and/or Körner intended "Čebyšev".

Remarkably, for any α we have a formula for K_m thanks to the three-term recursion ("Theorem 40.9"):

4. [Darboux-Christoffel formula] Find constants κ_n such that

$$K_n(x,y) = \kappa_n \frac{P_{n+1}(x)P_n(y) - P_n(x)P_{n+1}(y)}{x-y}$$

provided $x \neq y$. Check that your formula works for the Tchebychev polynomials T_n using the explicit formula $T_n(\cos\theta) = \cos n\theta$.

The families of orthogonal polynomials for which an explicit description is known include the *Gegenbauer polynomials*, which are orthogonal with respect to the inner product

$$(f,g)_c := \int_{-1}^{1} f(x)\, g(x)\, (1-x^2)^c \, dx$$

(where c is a real parameter greater than -1). Thus they generalize the polynomials of Tchebychev ($c = -1/2$) and Legendre ($c = 0$). There are various formulas and approaches for obtaining the Gegenbauer polynomials. We give here one that works particularly cleanly for the case $c = 1$, and is also relevant to problem B-2 on the 1999 Putnam examination:

5. Let $\mathcal{P} = \mathbf{R}[x]$ be the set of polynomials in x with real coefficients considered as a real vector space, and let $A_1 : \mathcal{P} \to \mathcal{P}$ be the linear operator defined by

$$(A_1 P)(x) = \frac{d^2}{dx^2}[(x^2-1)P(x)].$$

Prove that A_1 is self-adjoint with respect to $(\cdot,\cdot)_1$. Find, for each $n = 0, 1, 2, \ldots$, a real λ_n such that some polynomial u_n of degree n is a λ_n eigenvalue of A_1. Show that the λ_n for different n are distinct. Conclude that u_n are orthogonal polynomials with respect to $(\cdot,\cdot)_1$. Explain the relevance of this to Putnam 1999:B2.[2]

6. Generalize A_1 to a differential operator A_c whose eigen-polynomials are orthogonal polynomials with respect to $(\cdot,\cdot)_c$. Verify directly that the T_n are eigen-polynomials of $A_{-1/2}$ with the appropriate eigenvalues.

[2] The problem statement was: "Let $P(x)$ be a polynomial of degree n such that $P(x) = Q(x)P''(x)$, where $Q(x)$ is a quadratic polynomial and $P''(x)$ is the second derivative of $P(x)$. Show that if $P(x)$ has at least two distinct roots then it must have n distinct roots."

7. If the u_n in problems 5, 6 are chosen to be monic, what is (u_n, u_n), and what is the three-term recurrence they satisfy? Can you generalize Lemma 40.7 in Körner (pages 188–190) to Gegenbauer polynomials for arbitrary c?

8. (Laguerre polynomials.) Let $\mathcal{P}$ be the $\mathbf{R}$-vector space of polynomials, equipped with the scalar product

$$(P, Q) := \int_0^\infty P(x)Q(x)e^{-x}dx.$$

Define polynomials $L_n \in \mathcal{P}$ of degree n ($n = 0, 1, 2, \ldots$) by the generating function

$$\sum_{n=0}^{\infty} L_n(x)y^n = \frac{e^{xy/(y-1)}}{1-y}.$$

Show that these polynomials form an orthonormal set in $\mathcal{P}$. [You are granted special dispensation to manipulate the power series and integrals formally without worrying about convergence.] Compute L_0, L_1, L_2 and verify that they are indeed orthonormal. Can you prove that $\{e^{-x/2}L_n(x)\}_{n=0}^{\infty}$ is an ontb for $L_2([0, \infty))$? (It is known that this is true, but I haven't seen a simple/nice proof using ideas we have developed thus far.)

This problem set is due Monday, April 21 in class.