Math 55c: Honors Intermediate Advanced Remedial Calculus

FINAL EXAMINATION

55 July 1996

DON'T PANIC![1] Generous partial credit is available. The problems are ordered roughly in random order of difficulty.

SOLVE FIVE OF THE FOLLOWING FOUR PROBLEMS:

1. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ define $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ and analytically continue this to a function on the entire complex plane. Find, with proof, all the zeroes of $\zeta(s)$.[2] [HINT: Try to obtain a relationship between ζ, Γ, and ξ where ξ is something you must define.] Also prove in 55 different ways that $\zeta(2) = \frac{\pi^2}{6}$. For extra credit, find $\zeta(55)$.

2. Prove that if x, y, z, and n are integers with $n \geq 3$ and $x^n + y^n = z^n$ then $xyz = 0$. [HINT: Think of linear equations. Then think of quadratic equations. Then generalize!]

3. Carefully name the mathematician after whom the polynomials defining $\cos(nx)$ in terms of $\cos(x)$ are named. [HINT: You listen to Čaykovskiy, don't you?]

[1] D. Adams, *The Hitchhiker's Guide to the Galaxy.*

[2] Recall the definition of absolut convergence from Monday's class:

The natural numbers $\mathbb{N}$**:** with unity $1_{\mathbb{N}}$ and successor function $s_{\mathbb{N}}$.

$$\forall\alpha[[\alpha\in s_{\mathbb{N}}]\Rightarrow[\exists\beta\exists\gamma[\beta\in\mathbb{N}\wedge\gamma\in\mathbb{N}\wedge\{\{\gamma\},\{\beta,\gamma\}\}=\alpha]]]\wedge\forall\beta[[\beta\in\mathbb{N}]\Rightarrow[\exists\gamma[\{\{\gamma\},\{\beta,\gamma\}\}\in s_{\mathbb{N}}]]]\wedge\forall\beta\forall\gamma\forall\delta[[\{\{\gamma\},\{\beta,\gamma\}\}\in s_{\mathbb{N}}\wedge\{\{\delta\},\{\beta,\delta\}\}\in s_{\mathbb{N}}]\Rightarrow[\gamma=\delta]]\wedge 1_{\mathbb{N}}\in\mathbb{N}\wedge\forall\alpha[\neg[\{\{1_{\mathbb{N}}\},\{\alpha,1_{\mathbb{N}}\}\}\in s_{\mathbb{N}}]]\wedge\forall\alpha\forall\beta\forall\gamma[[\{\{\gamma\},\{\alpha,\gamma\}\}\in s_{\mathbb{N}}\wedge\{\{\gamma\},\{\beta,\gamma\}\}\in s_{\mathbb{N}}]\Rightarrow[\alpha=\beta]]\wedge\forall\zeta[[\forall\eta[[\eta\in\zeta]\Rightarrow[\eta\in\mathbb{N}]]\wedge 1_{\mathbb{N}}\in\zeta\wedge\forall\alpha\forall\beta[[\alpha\in\zeta\wedge\{\{\beta\},\{\alpha,\beta\}\}\in s_{\mathbb{N}}]\Rightarrow[\beta\in\zeta]]]\Rightarrow[\zeta=\mathbb{N}]]$$

The real numbers $\mathbb{R}$**:** with addition $+_{\mathbb{R}}$, zero $0_{\mathbb{R}}$, negation $-_{\mathbb{R}}$, multiplication $\times_{\mathbb{R}}$, unity $1_{\mathbb{R}}$, inverse ${}^{-1}{}_{\mathbb{R}}$, and positive subset $\mathbb{R}_+$.

$$\forall\alpha[[\alpha\in+_{\mathbb{R}}]\Rightarrow[\exists\beta\exists\gamma\exists\delta[\beta\in\mathbb{R}\wedge\gamma\in\mathbb{R}\wedge\delta\in\mathbb{R}\wedge\{\{\delta\},\{\{\beta,\gamma\},\delta\}\}=\alpha]]]\wedge\forall\beta\forall\gamma[[\beta\in\mathbb{R}\wedge\gamma\in\mathbb{R}]\Rightarrow[\exists\delta[\{\{\delta\},\{\{\beta,\gamma\},\delta\}\}\in+_{\mathbb{R}}]]]\wedge\forall\beta\forall\gamma\forall\delta\forall\epsilon[[\{\{\delta\},\{\{\beta,\gamma\},\delta\}\}\in+_{\mathbb{R}}\wedge\{\{\epsilon\},\{\{\beta,\gamma\},\epsilon\}\}\in+_{\mathbb{R}}]\Rightarrow[\delta=\epsilon]]\wedge 0_{\mathbb{R}}\in\mathbb{R}\wedge\forall\alpha[[\alpha\in\mathbb{R}]\Rightarrow[\{\{\alpha\},\{\{0_{\mathbb{R}},\alpha\},\alpha\}\}\in+_{\mathbb{R}}]]\wedge\forall\alpha[[\alpha\in-_{\mathbb{R}}]\Rightarrow[\exists\beta\exists\gamma[\beta\in\mathbb{R}\wedge\gamma\in\mathbb{R}\wedge\{\{\gamma\},\{\beta,\gamma\}\}=\alpha]]]\wedge\forall\beta[[\beta\in\mathbb{R}]\Rightarrow[\exists\gamma[\{\{\gamma\},\{\beta,\gamma\}\}\in-_{\mathbb{R}}]]]\wedge\forall\beta\forall\gamma\forall\delta[[\{\{\gamma\},\{\beta,\gamma\}\}\in-_{\mathbb{R}}\wedge\{\{\delta\},\{\beta,\delta\}\}\in-_{\mathbb{R}}]\Rightarrow[\gamma=\delta]]\wedge\forall\alpha\forall\beta[[\{\{\beta\},\{\alpha,\beta\}\}\in-_{\mathbb{R}}]\Rightarrow[\{\{0_{\mathbb{R}}\},\{\{\alpha,\beta\},0_{\mathbb{R}}\}\}\in+_{\mathbb{R}}]]\wedge\forall\alpha\forall\beta\forall\gamma\forall\delta\forall\epsilon\forall\zeta\forall\eta[[\{\{\delta\},\{\{\beta,\gamma\},\delta\}\}\in+_{\mathbb{R}}\wedge\{\{\epsilon\},\{\{\alpha,\delta\},\epsilon\}\}\in\times_{\mathbb{R}}\wedge\{\{\zeta\},\{\{\alpha,\beta\},\zeta\}\}\in\times_{\mathbb{R}}\wedge\{\{\eta\},\{\{\alpha,\gamma\},\eta\}\}\in\times_{\mathbb{R}}]\Rightarrow[\{\{\epsilon\},\{\{\zeta,\eta\},\epsilon\}\}\in+_{\mathbb{R}}]]\wedge\exists\alpha\exists\beta[\forall\gamma[[\gamma\in\alpha]\Rightarrow[\gamma\in\mathbb{R}]]\wedge\neg[0_{\mathbb{R}}\in\alpha]\wedge\forall\gamma[[\gamma\in\mathbb{R}\wedge\neg[\gamma=0_{\mathbb{R}}]]\Rightarrow[\gamma\in\alpha]]\wedge\forall\gamma[[\gamma\in\beta]\Rightarrow[\gamma\in\times_{\mathbb{R}}]]\wedge\forall\gamma[[\gamma\in\mathbb{R}]\Rightarrow[\{\{0_{\mathbb{R}}\},\{\{0_{\mathbb{R}},\gamma\},0_{\mathbb{R}}\}\}\in\times_{\mathbb{R}}]]\wedge\forall\delta[[\delta\in\times_{\mathbb{R}}\wedge\neg[\delta\in\beta]]\Rightarrow[\exists\gamma[\gamma\in\mathbb{R}\wedge\{\{0_{\mathbb{R}}\},\{\{0_{\mathbb{R}},\gamma\},0_{\mathbb{R}}\}\}=\delta]]]\wedge\forall\epsilon[[\epsilon\in\beta]\Rightarrow[\exists\zeta\exists\eta\exists\theta[\zeta\in\alpha\wedge\eta\in\alpha\wedge\theta\in\alpha\wedge\{\{\theta\},\{\{\zeta,\eta\},\theta\}\}=\epsilon]]]\wedge\forall\zeta\forall\eta[[\zeta\in\alpha\wedge\eta\in\alpha]\Rightarrow[\exists\theta[\{\{\theta\},\{\{\zeta,\eta\},\theta\}\}\in\beta]]]\wedge\forall\zeta\forall\eta\forall\theta\forall\iota[[\{\{\theta\},\{\{\zeta,\eta\},\theta\}\}\in\beta\wedge\{\{\iota\},\{\{\zeta,\eta\},\iota\}\}\in\beta]\Rightarrow[\theta=\iota]]\wedge 1_{\mathbb{R}}\in\alpha\wedge\forall\epsilon[[\epsilon\in\alpha]\Rightarrow[\{\{\epsilon\},\{\{1_{\mathbb{R}},\epsilon\},\epsilon\}\}\in\beta]]\wedge\forall\epsilon[[\epsilon\in{}^{-1}{}_{\mathbb{R}}]\Rightarrow[\exists\zeta\exists\eta[\zeta\in\alpha\wedge\eta\in\alpha\wedge\{\{\eta\},\{\zeta,\eta\}\}=\epsilon]]]\wedge\forall\zeta[[\zeta\in\alpha]\Rightarrow[\exists\eta[\{\{\eta\},\{\zeta,\eta\}\}\in{}^{-1}{}_{\mathbb{R}}]]]\wedge\forall\zeta\forall\eta\forall\theta[[\{\{\eta\},\{\zeta,\eta\}\}\in{}^{-1}{}_{\mathbb{R}}\wedge\{\{\theta\},\{\zeta,\theta\}\}\in{}^{-1}{}_{\mathbb{R}}]\Rightarrow[\eta=\theta]]\wedge\forall\epsilon\forall\zeta[[\{\{\zeta\},\{\epsilon,\zeta\}\}\in{}^{-1}{}_{\mathbb{R}}]\Rightarrow[\{\{1_{\mathbb{R}}\},\{\{\epsilon,\zeta\},1_{\mathbb{R}}\}\}\in\beta]]]\wedge\forall\alpha[[\alpha\in\mathbb{R}_+]\Rightarrow[\alpha\in\mathbb{R}]]\wedge\forall\alpha\forall\beta\forall\gamma[[\alpha\in\mathbb{R}_+\wedge\beta\in\mathbb{R}_+\wedge\{\{\gamma\},\{\{\alpha,\beta\},\gamma\}\}\in+_{\mathbb{R}}]\Rightarrow[\gamma\in\mathbb{R}_+]]\wedge\forall\alpha\forall\beta\forall\gamma[[\alpha\in\mathbb{R}_+\wedge\beta\in\mathbb{R}_+\wedge\{\{\gamma\},\{\{\alpha,\beta\},\gamma\}\}\in\times_{\mathbb{R}}]\Rightarrow[\gamma\in\mathbb{R}_+]]\wedge\forall\alpha[[\alpha\in\mathbb{R}]\Rightarrow[[\alpha=0_{\mathbb{R}}\wedge\neg[\alpha\in\mathbb{R}_+]\wedge\neg[\exists\beta[\{\{\beta\},\{\alpha,\beta\}\}\in-_{\mathbb{R}}\wedge\beta\in\mathbb{R}_+]]]\vee[\neg[\alpha=0_{\mathbb{R}}]\wedge\alpha\in\mathbb{R}_+\wedge\neg[\exists\beta[\{\{\beta\},\{\alpha,\beta\}\}\in-_{\mathbb{R}}\wedge\beta\in\mathbb{R}_+]]]\vee[\neg[\alpha=0_{\mathbb{R}}]\wedge\neg[\alpha\in\mathbb{R}_+]\wedge\exists\beta[\{\{\beta\},\{\alpha,\beta\}\}\in-_{\mathbb{R}}\wedge\beta\in\mathbb{R}_+]]]]\wedge\forall\alpha[[\forall\zeta[[\zeta\in\alpha]\Rightarrow[\zeta\in\mathbb{R}]]\wedge\exists\beta[\beta\in\alpha]\wedge\exists\beta[\forall\delta[[\delta\in\alpha]\Rightarrow[\delta=\beta\vee\exists\epsilon[\{\{\beta\},\{\{\delta,\epsilon\},\beta\}\}\in+_{\mathbb{R}}\wedge\epsilon\in\mathbb{R}_+]]]]]\Rightarrow[\exists\beta[\forall\delta[[\delta\in\alpha]\Rightarrow[\delta=\beta\vee\exists\epsilon[\{\{\beta\},\{\{\delta,\epsilon\},\beta\}\}\in+_{\mathbb{R}}\wedge\epsilon\in\mathbb{R}_+]]]\wedge\forall\gamma[[\forall\delta[[\delta\in\alpha]\Rightarrow[\delta=\gamma\vee\exists\epsilon[\{\{\gamma\},\{\{\delta,\epsilon\},\gamma\}\}\in+_{\mathbb{R}}\wedge\epsilon\in\mathbb{R}_+]]]\wedge\neg[\beta=\gamma]]\Rightarrow[\exists\epsilon[\{\{\gamma\},\{\{\beta,\epsilon\},\gamma\}\}\in+_{\mathbb{R}}\wedge\epsilon\in\mathbb{R}_+]]]]]]$$

Metric space X**:** with metric d.

$$\forall\beta[[\beta\in d]\Rightarrow[\exists\gamma\exists\delta\exists\epsilon[\gamma\in X\wedge\delta\in X\wedge\epsilon\in\mathbb{R}\wedge\{\{\epsilon\},\{\{\gamma,\delta\},\epsilon\}\}=\beta]]]\wedge\forall\gamma\forall\delta[[\gamma\in X\wedge\delta\in X]\Rightarrow[\exists\epsilon[\{\{\epsilon\},\{\{\gamma,\delta\},\epsilon\}\}\in d]]]\wedge\forall\gamma\forall\delta\forall\epsilon\forall\zeta[[\{\{\epsilon\},\{\{\gamma,\delta\},\epsilon\}\}\in d\wedge\{\{\zeta\},\{\{\gamma,\delta\},\zeta\}\}\in d]\Rightarrow[\epsilon=\zeta]]\wedge\forall\beta\forall\gamma\forall\eta[[\{\{\eta\},\{\{\beta,\gamma\},\eta\}\}\in d]\Rightarrow[\eta\in\mathbb{R}_+\vee\eta=0_{\mathbb{R}}]]\wedge\forall\beta\forall\gamma[[\{\{0_{\mathbb{R}}\},\{\{\beta,\gamma\},0_{\mathbb{R}}\}\}\in d]\Rightarrow[\beta=\gamma]]\wedge\forall\beta[\{\{0_{\mathbb{R}}\},\{\{\beta,\beta\},0_{\mathbb{R}}\}\}\in d]\wedge\forall\beta\forall\gamma\forall\delta\forall\epsilon\forall\zeta\forall\eta\forall\theta[[\{\{\epsilon\},\{\{\beta,\gamma\},\epsilon\}\}\in d\wedge\{\{\zeta\},\{\{\gamma,\delta\},\zeta\}\}\in d\wedge\{\{\eta\},\{\{\beta,\delta\},\eta\}\}\in d\wedge\{\{\theta\},\{\{\epsilon,\zeta\},\theta\}\}\in+_{\mathbb{R}}]\Rightarrow[[\exists\alpha[\{\{\theta\},\{\{\eta,\alpha\},\theta\}\}\in+_{\mathbb{R}}\wedge\alpha\in\mathbb{R}_+]\vee\theta=\eta]]]$$

Sequence f**:** in the metric space X.

$$\forall\beta[[\beta\in f]\Rightarrow[\exists\gamma\exists\delta[\gamma\in\mathbb{N}\wedge\delta\in X\wedge\{\{\delta\},\{\gamma,\delta\}\}=\beta]]]\wedge\forall\gamma[[\gamma\in\mathbb{N}]\Rightarrow[\exists\delta[\{\{\delta\},\{\gamma,\delta\}\}\in f]]]\wedge\forall\gamma\forall\delta\forall\epsilon[[\{\{\delta\},\{\gamma,\delta\}\}\in f\wedge\{\{\epsilon\},\{\gamma,\epsilon\}\}\in f]\Rightarrow[\delta=\epsilon]]$$

Absolut convergence: of f in X with metric d.

$$\exists\beta[\forall\gamma[[\gamma\in\beta]\Rightarrow[\exists\delta\exists\epsilon[\delta\in\mathbb{N}\wedge\epsilon\in\mathbb{R}\wedge\{\{\epsilon\},\{\delta,\epsilon\}\}=\gamma]]]\wedge\forall\delta[[\delta\in\mathbb{N}]\Rightarrow[\exists\epsilon[\{\{\epsilon\},\{\delta,\epsilon\}\}\in\beta]]]\wedge\forall\delta\forall\epsilon\forall\zeta[[\{\{\epsilon\},\{\delta,\epsilon\}\}\in\beta\wedge\{\{\zeta\},\{\delta,\zeta\}\}\in\beta]\Rightarrow[\epsilon=\zeta]]\wedge\{\{0_{\mathbb{R}}\},\{1_{\mathbb{N}},0_{\mathbb{R}}\}\}\in\beta\wedge\forall\gamma\forall\delta\forall\epsilon\forall\zeta\forall\eta\forall\theta\forall\iota[[\{\{\delta\},\{\gamma,\delta\}\}\in s_{\mathbb{N}}\wedge\{\{\epsilon\},\{\gamma,\epsilon\}\}\in f\wedge\{\{\zeta\},\{\delta,\zeta\}\}\in f\wedge\{\{\eta\},\{\gamma,\eta\}\}\in\beta\wedge\{\{\iota\},\{\{\epsilon,\zeta\},\iota\}\}\in d\wedge\{\{\theta\},\{\{\eta,\iota\},\theta\}\}\in+_{\mathbb{R}}]\Rightarrow[\{\{\theta\},\{\delta,\theta\}\}\in\beta]]\wedge\exists\theta[\forall\gamma\forall\iota[[\{\{\iota\},\{\gamma,\iota\}\}\in\beta]\Rightarrow[\exists\alpha[\{\{\theta\},\{\{\iota,\alpha\},\theta\}\}\in+_{\mathbb{R}}\wedge\alpha\in\mathbb{R}_+]]]]]$$