Math 272y: Rational Lattices and their Theta Functions

30 September and 2 October 2019:
Positive-definite integral lattices generated by vectors of norm at most 2

We have seen several examples where, given the rank of a self-dual positive-definite lattice L, the modularity of θ_L lets us count the lattice vectors of norm 1 or 2. (We shall obtain more such examples once we consider the theta functions of integral lattices that need not be self-dual.) It turns out that there is a complete classification of lattices L_0 generated by their vectors of norm 1 and 2: such L_0 is the direct sum of lattices each isomorphic with $\mathbf{Z}$ or an ADE root lattice. In this chapter of the lecture notes, we give this classification and some invariants of the ADE lattices, and use this information to classify self-dual lattices of rank up to 15, and self-dual even ones of rank 16 (where so far we reached only rank 7 and 8 respectively).

Removing vectors of norm 1. Suppose first that L is an integral lattice (not necessarily positive-definite, and possibly degenerate), and v_0 is a lattice vector with $\langle v_0, v_0\rangle = 1$. Then v_0 generates a sublattice $\mathbf{Z}v_0$ of L isomorphic with $\mathbf{Z}$. We claim that this $\mathbf{Z}$ is a "direct summand": there is some other sublattice $L_1 \subset L$ such that $L = \mathbf{Z}v_0 \oplus L_1$. The reason is that the orthogonal projection $\pi_0 : L \otimes \mathbf{R} \to \mathbf{R}v_0$ is given by

$$\pi_0(v) = \frac{\langle v, v_0\rangle}{\langle v_0, v_0\rangle} v_0 = \langle v, v_0\rangle\, v_0,$$

and thus maps L to $\mathbf{Z}v_0$. Hence the complementary projection $\pi_1 = 1 - \pi_0$ takes any $v \in L$ to $v - \langle v, v_0\rangle\, v_0 \in L$, so if we set $L_1 := \pi_1(L)$ (which is also $\ker(\pi_0 : L \to \mathbf{Z}v_0)$) then $L = \mathbf{Z}v_0 \oplus L_1$ as claimed.

By induction it follows that if L contains a sublattice L_0 isomorphic to $\mathbf{Z}^k$ for some k then L_0 is a direct summand, i.e., there is some other sublattice $L_k \subset L$ such that $L = L_0 \oplus L_k$. This gives us an alternative argument to conclude the proof that if L has rank n and $N_1(L) = 2n$ then $L \cong \mathbf{Z}^n$ (as in the classification of positive-definite self-dual lattices of rank $n \leq 7$, or of any rank n and with minimal characteristic norm n). In general, if L is positive-definite of rank n with $2k$ vectors of norm 1, it is enough to describe the lattice L_k which has rank $n - k$ and *no* vectors of norm 1, and then recover L as $\mathbf{Z}^k \oplus L_k$. If we assume that L is generated by vectors of norm at most 2, then L_k is generated by vectors of norm *exactly* 2. Such lattices are called *root lattices*, and are the topic of most of this week's lecture notes. For now we remark that such a lattice must be even, because it is integral and generated by vectors whose norm is the even number 2.

If L is also self-dual, then the same is true of L_k. We can then use the fact that θ_{L_k} is a modular form for Γ_+. For example:[1]

Proposition. *Let* L *be a positive-definite self-dual lattice of rank* n *with* $0 < n < 16$. *If* $N_1(L) = 0$ *then* $n \geq 8$ *and*

$$\theta_L = \theta_{\mathbf{Z}}^n - 2n\theta_{\mathbf{Z}}^{n-8}\Delta_+ = 1 + 2n(23 - n)q + O(q^{3/2}), \tag{1}$$

[1] This is again from my paper "Lattices and codes with long shadows" in *Math. Research Letters* **2** (1995), 643–651 (arXiv: math.NT/9906086), which lists all L satisfying the hypothesis. For now we can do this only for $n < 16$; once we obtain the classification of Niemeier lattices we'll be able to handle also the remaining possibilities with $16 \leq n \leq 23$.

so L has $2n(23-n)$ roots. More generally (1) holds for any n if $N_1(L)=0$ and L has minimal characteristic norm $n-8$; hence such a lattice has $n \le 23$.

Proof: Since $n < 16$ we know that θ_L is a linear combination of $\theta_{\mathbf{Z}}^n$ and $\theta_{\mathbf{Z}}^{n-8}\Delta_+$. The coefficient of $\theta_{\mathbf{Z}}^n$ is $N_0(L)=1$. The hypothesis $N_1(L)=0$ then determines the coefficient of $\theta_{\mathbf{Z}}^{n-8}\Delta_+$. This yields (1), from which we read off $N_2(L)=2n(23-n)$. Note that such a lattice must have minimal characteristic norm $n-8$: since $n<16$, the only possibilities are n and $n-8$, and we showed last week that n arises only for $\mathbf{Z}^n$. If we allow any n but require that the minimal characteristic norm equal $n-8$, then θ_L is a multiple of $\theta_{\mathbf{Z}}^{n-8}$, and again we deduce (1), from which $n \le 23$ follows because $N_2(L) \ge 0$. $\Box$

Of course if $\langle v_0, v_0 \rangle = -1$ then $\pi_0(v) = -\langle v, v_0\rangle\, v_0$ likewise proves that $\mathbf{Z}v$ is a direct summand; but we won't have much use for this because our main focus is on positive-definite lattices. More generally, if $L_0 \subset L$ is any *unimodular* sublattice (that is, a sublattice with $\operatorname{disc} L_0 = \pm 1$), then L_0 is a direct summand; it's rare that this can be applied for L_0 other than $\mathbf{Z}^k$ (and $\mathbf{Z}^{k_1} \oplus \mathbf{Z}^{k_2}\langle -1\rangle$ if we do not require positive-definiteness), but we shall use it here for $L_0 = E_8$, so we recite a proof. Let π_0 be the orthogonal projection from $L \otimes \mathbf{R}$ to $L_0 \otimes \mathbf{R}$, which exists because L_0 is nondegenerate; and let $1-\pi$ be the complementary projection. If $v \in L_0$ then $w \mapsto \langle v, w\rangle$ is a homomorphism from L_0 to $\mathbf{Z}$; under our hypothesis, L_0 is self-dual, so this homomorphism is represented by the inner product with some vector $v_0 \in L_0$. Then this v_0 is $\pi_0(v)$, so $\pi_0(L) = L_0$ and we obtain a complementary summand as $\pi_1(L) = (1-\pi_0)L$.

Root lattices: positive-definite integral lattices generated by vectors of norm 2. Now suppose v_0 is a vector of norm 2 in L. Such a vector is often called a *root vector* of L.[2] In general the orthogonal projection π_0 does not take L to $\mathbf{Z}v_0$, because of the denominator $\langle v_0, v_0\rangle = 2$ in the formula for π_0. However, the *reflection*

$$\rho_{v_0} = 1 - 2\pi_0 : v \mapsto v - 2\frac{\langle v, v_0\rangle}{\langle v_0, v_0\rangle} v_0 = v - \langle v, v_0\rangle\, v_0 \tag{2}$$

through the hyperplane perpendicular to v_0 does take L to L This makes tractable the classification of integral lattices generated by vectors of norm 2 (whereas we do not expect any nice description of lattices generated by vectors of norm N for any $N \ge 3$). Note that ρ_{v_0}, being a reflection, permutes the roots, and thus acts also on the *root sublattice* $R(L)$, which is the sublattice of L generated by its roots. Thus the same is true of the subgroup of $\operatorname{Aut}(L)$ generated by the root reflections ρ_{v_0}; this subgroup is called the *Weyl group* of L (and thus also of $R(L)$), denoted $W(L) = W(R(L))$.

For future reference we note that $W(L)$ acts trivially on L^*/L: if $v \in L^*$ then $\rho_{v_0}(v) \equiv v \bmod L$. As a consequence, if $\pm 1 \in W(L)$ then -1 acts trivially on L^*/L, so L^*/L has exponent 2 (or is trivial). This necessary condition turns out to be sufficient (it is satisfied by root lattices whose simple components are all A_1, D_n for even n, or E_n for $n=7$ or $n=8$).

[2] Warning: in some sources and contexts, a "root" is any primitive lattice vector v_0, not necessarily of norm 2, for which the reflection ρ_{v_0} takes L to L. (A lattice vector v is "primitive" if it is not of the form nv_1 for some lattice vector v_1 and integer $n \ge 2$.) For example, the vectors $(1,0)$ and $(0,1)$ in $\mathbf{Z}\langle c\rangle \oplus \mathbf{Z}\langle c'\rangle$ would be regarded as "roots" of that lattice, as would vectors of norm 6 in the A_2 lattice (which are the long roots of the G_2 root system). We require our roots to have norm 2.

If L_1 and L_2 are root lattices, then so is $L_1 \oplus L_2$, with $W(L_1 \oplus L_2) = W(L_1) \times W(L_2)$. A *simple* root lattice is one that cannot be written as $(L_1 \oplus L_2)$ for any nonzero root lattices L_1, L_2. The simple root lattices are completely classified: there are two infinite families A_n, D_n, contained in $\mathbf{Z}^{n+1}$ and $\mathbf{Z}^n$ respectively, and three exceptional simple root lattices E_6, E_7, E_8 which are not contained in any $\mathbf{Z}^N$. These lattices are remarkably ubiquitous: besides being basic building blocks of integral quadratic forms, they are fundamental to the theory of Lie algebras, and appear also in finite group theory, algebraic geometry, and elsewhere; if you pursue mathematics then you'll likely encounter the ADE classification before long, even if you never think about sphere packing or Niemeier lattices again. It is thus worth taking the time to know these lattices in some detail.

We recite without proof some basic structural results; though not obvious, they are well-known — see for instance Chapter V "Root Systems" of Serre's *Complex Semisimple Lie Algebras* for the proofs (keeping in mind that Serre allows root systems where not all roots have the same norm). Let L be a root lattice, and $\mathcal{R} = \{v \in L : \langle v, v \rangle = 2\}$ its *root system* of norm-2 vectors. Choose any linear functional $L \to \mathbf{R}$ not vanishes on any root; this partitions $\mathcal{R}$ into positive roots and negative roots. A positive root is said to be *simple* if it is not the sum of two positive roots. Let $\mathcal{B}$ be the set of simple roots; such a set is also called a *base* for $\mathcal{R}$. Then: $\mathcal{B}$ is a $\mathbf{Z}$-basis for L; every positive root is $\sum_{r \in \mathcal{B}} c_r r$ with each $c_r \geq 0$; and $W(L)$ acts simply transitively on the bases of $\mathcal{R}$.

If $r, r' \in \mathcal{B}$ then either $r = r'$ or $\langle r, r' \rangle \leq 0$, because the only other possibility is $\langle r, r' \rangle = 1$, and then $r - r'$ is a root that is neither positive nor negative. Hence $\langle r, r' \rangle$ is either 0 or -1. The *Coxeter-Dynkin diagram* of $\mathcal{R}$ is the undirected graph whose vertices are simple roots and whose edges are pairs $\{r, r'\}$ with $\langle r, r' \rangle = -1$. (This graph depends only on $\mathcal{R}$, not on the choice of base, because $W(L)$ acts transitively on bases and preserves inner products.) The graph is connected if and only if L is a simple root lattice. The Gram matrix of $\mathcal{B}$ is then $2I_n - A$ where A is the adjacency matrix of the Coxeter-Dynkin diagram; hence $2I_n - A$ is positive-definite. Remarkably this necessary condition is also sufficient: if a graph G has adjacency matrix A with $2I_n - A$ positive-definite, then G is the Coxeter-Dynkin diagrams of a root system, namely the roots of the even lattice with Gram matrix $2I_n - A$.

It is a standard exercise to find all G satisfying this condition. We may assume G is connected, because the condition holds for G if and only if it holds for each connected component of G. Then G must be a tree, because if some simple roots formed a cycle then their sum would have norm zero. If G is a path of length n then L is the A_n lattice; otherwise G has a vertex of degree at least 3. But the degree cannot exceed 3, because if a simple root r had four neighbors r_1, r_2, r_3, r_4 then $2r + \sum_{j=1}^{4} r_j$ would have norm zero. Likewise there cannot be more than one degree-3 vertex: since G is connected, any two such vertices would be connected by a path $r_0, r_1, \ldots, r_k$, and then we would obtain a norm-zero vector by adding to $\sum_{j=0}^{k} r_j$ the sum of the neighbors of r_0 and r_k other than r_1 and r_{k-1}. So, G is either a path or is obtained from three paths, say of lengths e_1, e_2, e_3, meeting at one vertex (which we count as part of the path, so each $e_j \geq 2$ and $n = e_1 + e_2 + e_3 - 2$). In the latter case we find that $2A_n - I$ is positive-definite if and only if $1/e_1 + 1/e_2 + 1/e_3 > 1$. As we noted in connection with spherical triangle groups,[3] this happens

[3]This connection is one manifestation of the celeberated and fruitful McKay correspondence, which alas would take us too far afield here.

if and only if e_1, e_2, e_3 are a permutation of either $2, 2, e$ for some $e \geq 2$ or $2, 3, e$ for one of $e = 3, 4, 5$. The $2, 2, e$ case yields D_n with $n = e + 2$, and the exceptional $2, 3, e$ yields E_n with $n = e + 3$.

If L is a root lattice (not necessarily simple) with base $\mathcal{B}$ then the reflections ρ_r with $r \in \mathcal{B}$ generate $W(L)$. For simple roots r, r' these reflections satisfy $(\rho_r \rho_{r'})^3 = 1$ if r, r' are adjacent in the Coxeter-Dynkin diagram, and $(\rho_r \rho_{r'})^2 = 1$ otherwise (in which case $\rho_r \rho_{r'} = \rho_{r'} \rho_r$); it is known that these relations, together with $\rho_r^2 = 1$ for each r, give a presentation of $W(L)$. The full automorphism group of L contains $W(L)$ as a normal subgroup, with quotient equal to the automorphism group of the Coxeter-Dynkin diagram. In the case of a simple root lattice, this automorphism group is trivial for A_1, E_7, E_8, and otherwise has order 2 except for the D_4 diagram which has automorphism group S_3.

It is known that a finite subgroup $G \subset \mathrm{O}_n(\mathbf{R})$ is generated by reflections if and only if the subring of $\mathbf{R}[x_1, \ldots, x_n]$ invariant under G is a polynomial ring, and then this ring is generated by homogeneous polynomials whose degrees $d_1, \ldots, d_n$ satisfy $\#G = \prod_{j=1}^{n} d_j$.[4] This generalizes the familiar theory of symmetric functions, for which G is the group of permutation matrices and the invariants can be chosen to have each $d_j = j$. For $G = W(L)$ the invariant ring and the d_j are often useful; we shall need only the smallest two d_j for a simple root lattice L. In this case $W(L)$ acts irreducibly on $L \otimes \mathbf{C}$, so the invariant polynomials of degree 1 and 2 have dimension 0 and 1 respectively (the norm $x \mapsto \langle x, x \rangle$ generates the quadratic invariants). The next invariant degree is $3, 4, 5, 6, 8$ for $L = A_n, D_n, E_6, E_7, E_8$ respectively.

We next tabulate some key information about each of the simple root lattices $L = A_n, D_n, E_n$, and show how to recover the classification of self-dual lattices in $\mathbf{R}^n$ for n up to 16. We then give a more detailed description of each case in turn. The table lists, for each L:

- The discriminant of L and the structure of its discriminant group L^*/L.
- The minimal norms of the cosets of L in L^*. We denote the minimal norm of the coset c by $\tilde{q}(c)$ because it is a lift of the discriminant form q from $\mathbf{Q}/2\mathbf{Z}$ to $\mathbf{Q}$. (For $L = A_n$, we identify L^*/L with $\mathbf{Z}/(n+1)\mathbf{Z}$, and then label each coset by the corresponding integer $m \in [0, n+1]$; for D_n, see the description below.) The values of this $\tilde{q}$ also appear in the formula for the canonical height of a point (or section) of an elliptic curve (or surface).
- The number $N_2(L)$ of roots of L, and the Coxeter number $h = N_2(L)/n$ (which also arises as the largest d_j and elsewhere in the theory).
- The Weyl group $W(L)$ for $L = A_n$ and D_n, and the d_j for $L = E_n$.

[4]Likewise for finite subgroups of $\mathrm{U}_n(\mathbf{C})$ generated by reflections (linear transformations ρ such that $1 - \rho$ has rank 1) and their action on $\mathbf{C}[x_1, \ldots, x_n]$. This is the theorem of Chevalley and Shephard-Todd; Shephard and Todd proved in it 1954 by giving a complete classification of irreducible complex reflection groups, and Chevalley gave a uniform proof in 1955.

L	$\operatorname{disc}(L)$	L^*/L	$\tilde{q}$	$N_2(L)$	h	$W(L)$ or d_j
$A_n (n \geq 1)$	$n+1$	$\mathbf{Z}/(n+1)\mathbf{Z}$	$\frac{m(n+1-m)}{n+1}$	n^2+n	$n+1$	S_{n+1}
$D_n (n \geq 4)$	4	$\begin{cases} (\mathbf{Z}/2\mathbf{Z})^2, & n \text{ even} \\ \mathbf{Z}/4\mathbf{Z}, & n \text{ odd} \end{cases}$	$0, 1, \frac{n}{4}, \frac{n}{4}$	$2(n^2-n)$	$2n-2$	$(\mathbf{Z}/2\mathbf{Z})^{n-1} \rtimes S_n$
E_6			$0, \frac{4}{3}, \frac{4}{3}$	72	12	$2, 5, 6, 8, 9, 12$
E_7	$9-n$	$\mathbf{Z}/(9-n)\mathbf{Z}$	$0, \frac{3}{2}$	126	18	$2, 6, 8, 10, 12, 14, 18$
E_8			0	240	30	$2, 8, 12, 14, 18, 20, 24, 30$

For example, if $n \leq 16$ then $h \leq 30$ with equality only for $L = E_8$ and D_{16}. Thus any even positive-definite lattice of rank $n \leq 16$ has at most $30n$ roots, with equality if and only if the lattice contains E_8, E_8^2, or D_{16} with finite index. But we showed last week that if the lattice is also self-dual then it has theta function $\mathsf{E}_4^{n/8}$ and thus has $30n$ roots. This proves that the only such lattices are E_8 for $n = 8$ and E_8^2, D_{16}^+ for $n = 16$.

Suppose now that L is positive-definite and self-dual but not necessarily even. If $n < 16$ then we can write $L = \mathbf{Z}^k \oplus L_k$ where L_k is also positive-definite and self-dual, of rank $n-k < 16$, which has no vectors of norm 1. Hence $N_2(L_k)$ is given by (1). If $L_k = (0)$ then $L \cong \mathbf{Z}^n$. Otherwise, if L_k contains E_8 then this E_8 is a direct summand, and then the complement has rank $n-k-8 < 8$ and is thus (0), so $L \cong \mathbf{Z}^{n-8} \oplus E_8$. The remaining cases are handled by our table. For example, if $n \leq 12$ then $N_2(L) = 2n(23-n) \geq 22n$, but the only simple root lattices of rank at most 12 that have $h \geq 22$ are E_8 and D_{12}, and we have eliminated E_8. Thus $L \supset D_{12}$, and then $[L : D_{12}] = 2$ and we conclude (as we did for $n = 8$ and $n = 16$) that $L \cong D_{12}^+$. We can go a bit further by listing, for each of $n = 13, 14, 15$, all root lattices of rank at most n with exactly $2n(23-n)$ roots. There are none for $n = 13$; for $n = 14$, only E_7^2; and for $n = 15$, either A_{15} or $A_4 D_{11}$. The last of these is not possible, because then L would contain $A_4 D_{11}$ with finite index, but this index would have to be a square root of $\operatorname{disc}(A_4 D_{11}) = 5 \cdot 4 = 20$, which is not a square. Each of the remaining cases yields a unique lattice L, containing the root lattice with index 2 (for $n = 14$) or 4 (for $n = 15$).

Once we develop the gluing method and the classification of the Niemeier lattices we'll be able to give an alternative explanation of our list of self-dual lattices of rank up to 16 and to extend it to $n < 24$.

We conclude this chapter of the notes with further details about the simple root lattices A_n, D_n, and E_n.

The root lattice A_n is the "trace-zero" slice of $\mathbf{Z}^{n+1}$:

$$A_n = \{(x_1, \ldots, x_{n+1}) \in \mathbf{Z}^{n+1} : \sum_{j=1}^{n+1} x_j = 0\}, \tag{3}$$

with the usual inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=1}^{n+1} x_j y_j$. Letting $e_1, \ldots, e_{n+1}$ be the standard unit vectors of $\mathbf{Z}^{n+1}$, we see that the roots of A_n are the $n^2 + n$ vectors $e_j - e_k$ with $j \neq k$. We claim that $W(A_n)$ is the group S_{n+1} of coordinate permutations of $\mathbf{Z}^{n+1}$, acting on the trace-zero subspace. Indeed the root $e_j - e_k$ yields the reflection that acts on the coordinates by the simple transposition $(j\,k)$, and these generate S_{n+1}. The full automorphism group of A_n is $\{\pm 1\} \times S_{n+1}$, except for $n = 1$ when the nontrivial element of S_2 already acts by -1 (and indeed in this case the Coxeter-Dynkin diagram consists of a single vertex and thus has no automorphisms). We can use $\{e_j - e_k : j < k\}$ for the positive roots, and then the simple roots are $\alpha_j = e_j - e_{j+1}$ $(1 \le j \le n)$, with

$$\langle \alpha_j, \alpha_{j'} \rangle = \begin{cases} 2, & \text{if } j' = j; \\ -1, & \text{if } |j' - j| = 1; \\ 0, & \text{if } |j' - j| > 1. \end{cases} \tag{4}$$

We can then compute that the corresponding Gram matrix has determinant $n + 1$; thus $\operatorname{disc} A_n = n+1$. The automorphism of the Coxeter-Dynkin diagram comes from the linear transformation taking $(x_1, x_2, \ldots, x_{n+1})$ to $(-x_{n+1}, -x_n, \ldots, -x_1)$ (which as expected is the identity for $n = 1$). The dual lattice is the orthogonal projection of $\mathbf{Z}^{n+1}$ to $A_n \otimes \mathbf{R}$, and consists of all $(x_1, \ldots, x_{n+1})$ with $\sum_{j=1}^{n+1} x_j = 0$ and $x_j - x_k \in \mathbf{Z}$ for all j, k (whence $x_j \in (n+1)^{-1}\mathbf{Z}$ for each j). The discriminant group A_n^*/A_n is cyclic, generated by the coset of $e_1 - (n+1)^{-1} \sum_{j=1}^{n+1} e_j$. For each integer m with $0 \le m \le n+1$, the m-th multiple of this coset has minimal norm $m(n+1-m)/(n+1)$, attained by the $\binom{n+1}{m}$ permutations of $\sum_{j=1}^{m} e_j - \frac{m}{n+1} \sum_{j=1}^{n+1} e_j$; these are the vectors with m coordinates $(n+1-m)/(n+1)$ and the remaining coordinates $-m/(n+1)$.

If L is a lattice such that $A_n \subseteq L \subseteq A_n^*$ with $[L : A_n] = k$, then $k \mid n+1$, and then there is a unique such lattice L (the preimage in A_n^* of the k-element subgroup of A_n^*/A_n); we call this lattice A_n^{+k}. Of course A_n^{+1} is just A_n itself (and A_n^* is $A_n^{+(n+1)}$). Also, for $k = 2$, we'll often abbreviate A_n^{+2} to A_n^+, as we did for $D_8^+ (= E_8)$, D_{12}^+, and D_{16}^+. Now $\operatorname{disc}(A_n^{+k}) = \operatorname{disc}(A_n)/[A_n^{+k} : A_n]^2 = (n+1)/k^2$; thus if A_n^{+k} is integral then $k^2 \mid n+1$, and we readily check that this necessary condition is also sufficient. In this case A_n^{+k} is even unless $n+1$ is even but $(n+1)/k^2$ is odd. For example, $A_3^{+2} = A_3^+$ is an odd lattice of discriminant $4/2^2 = 1$, so it must be isomorphic with $\mathbf{Z}^3$, and indeed it has orthonormal generators

$$\frac{1}{2}(e_1 + e_2 - e_3 - e_4), \quad \frac{1}{2}(e_1 - e_2 + e_3 - e_4), \quad \frac{1}{2}(e_1 - e_2 - e_3 + e_4). \tag{5}$$

The next examples are A_7^+, an even lattice of discriminant 2 isomorphic with the root lattice E_7, and A_8^{+3}, an even lattice of discriminant 1 and thus isomorphic with E_8. We can check this by counting roots: A_7^+ has $N_2(A_7) + \binom{8}{4} = 56 + 70 = 126$ roots, which must form an E_7 system (all other root lattices of rank ≤ 7 have Coxeter number < 18), and A_7^+ can be no larger than E_7 because $\operatorname{disc} E_7 = 2$ is squarefree. Likewise A_8^{+3} has $N_2(A_8) + \binom{9}{3} + \binom{9}{6} = 72 + 84 + 84 = 240$ roots, same as E_8. This is the last case where A_n^{+k} is integral but has vectors of norm ≤ 2 that are not already in A_n; the next two such lattices of discriminant 1 are A_{15}^{+4}, the unique self-dual lattice in $\mathbf{R}^{15}$ with no vectors of norm 1, and the Niemeier lattice A_{24}^{+5}.

It is well known that the S_{n+1}-invariant subring of $\mathbf{C}[x_1, \ldots, x_{n+1}]$ is polynomial, with one generator in degree d for each $d = 1, \ldots, n+1$; for example, the elementary symmetric functions of the x_i are generators. Since the A_n slice is the zero-locus of the degree-1 generator, the invariant degrees of A_n are the n integers $d \in [2, n+1]$. As expected their product is $(n+1)! = \#W(A_n)$.

There is a formula for θ_{A_n} and the theta series of the cosets of A_n in A_n^*, but it is somewhat unwieldy: one first takes the direct sum with $\mathbf{Z}\langle 1/(n+1)\rangle$, finds the theta series of *that* lattice by writing it as a union of $n+1$ translates of $\mathbf{Z}^{n+1}$, and then divides by $\theta_{\mathbf{Z}\langle 1/(n+1)\rangle}$. See (56,57) on page 110 of SPLAG for the formula (expressed in terms of a Jacobi theta function). We shall give simpler formulas for a few small n later in the course.

For small n, the lattices A_n are familiar: $A_1 \cong \mathbf{Z}\langle 2\rangle$, and A_2 is the triangular (a.k.a. hexagonal) lattice, scaled so its six minimal vectors have norm 2. For A_3, see the remarks on D_3 in the next section.

The root lattice D_n is the "checkerboard lattice", which is the even sublattice of $\mathbf{Z}^n$:

$$D_n = \{(x_1, \ldots, x_n) \in \mathbf{Z}^n : \sum_{j=1}^{n} x_j \equiv 0 \bmod 2\}, \tag{6}$$

with $\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{j=1}^{n} x_j y_j$. Thus $\operatorname{disc} D_n = [\mathbf{Z}^n : D_n]^2 \operatorname{disc} \mathbf{Z}^n = 2^2 \cdot 1 = 4$. The dual lattice D_n^* is the disjoint union of $\mathbf{Z}^n$ with $(\mathbf{Z}+\frac{1}{2})^n$. The structure of the discriminant group D_n^*/D_n depends on the parity of n, because $\sum_{j=1}^{n} e_j \in D_n$ iff n is even, so the D_n coset of the dual vector $\frac{1}{2}\sum_{j=1}^{n} e_j$ has order 2 or 4 according as n is even or odd; the discriminant group is $(\mathbf{Z}/2\mathbf{Z})^2$ in the former case, and $\mathbf{Z}/4\mathbf{Z}$ in the latter. Either way, each coset of half-integral vectors has minimal norm $n/4$, attained by 2^{n-1} of the 2^n vectors $\frac{1}{2}\sum_{j=1}^{n} \pm e_j$, and the remaining coset has minimal norm 1, attained by the $2n$ vectors $\pm e_j$ which are the minimal vectors of $\mathbf{Z}^n$.

The roots of D_n are the $2n(n-1)$ vectors $\pm e_j \pm e_k$ with $j \neq k$. For the simple roots we can take $e_j - e_{j+1}$ $(1 \leq j \leq n-1)$, together with $e_{n-1} + e_n$; then the degree-3 vertex of the Coxeter-Dynkin graph is the root $e_{n-2} - e_{n-1}$, and the graph automorphism negates the n-th coordinate. The reflection $\rho_{\pm e_j \pm e_k}$ switches coordinates j and k, and also negates both of them for the roots $\pm(e_j + e_k)$. Thus $W(D_n)$ is the index-2 subgroup of the hyperoctahedral group $\operatorname{Aut}(\mathbf{Z}^n) = \{\pm 1\}^n \rtimes S_n$ that negates an even number of coordinates. The hyperoctahedral group is itself a reflection group, with invariant ring generated by the elementary symmetric functions of the x_j^2. Replacing the last symmetric function $\prod_{j=1}^{n} x_j^2$ by its square root $\prod_{j=1}^{n} x_j$ gives the invariants of $W(D_n)$. Thus the invariant degrees are $2d$ for $1 \leq d \leq n-1$ together with n (so n appears twice if it is even). Their product is $2^{n-1} n! = \#W(D_n)$ as expected.

While D_n can be defined for all $n \geq 1$, one usually assumes $n \geq 4$ so as to list each irreducible root lattice only once. The lattice D_3 is isomorphic with A_3; for example, A_3 consists of the even linear combinations of the orthonormal vectors in (5). This identifies A_3 with the "face-centered cubic lattice". The lattice D_2 is reducible, being isomorphic with $A_1^2 = \mathbf{Z}^2\langle 2\rangle$, with orthogonal generators $e_1 \pm e_2$. The lattice D_1 is not even a root lattice: it is $2\mathbf{Z} = \mathbf{Z}\langle 4\rangle$, and has no roots at all. In each case, though, the identification with A_3, A_1^2, or $\mathbf{Z}\langle 4\rangle$ is consistent with the minimal norms $1, n/4, n/4$ of the nontrivial cosets of D_n in D_n^*, and also with the Weyl groups of A_3 and A_1^2; notably, $W(D_3) \cong W(A_3)$ recovers the isomorphism $S_4 = (\mathbf{Z}/2\mathbf{Z})^2 \rtimes S_3$. This also means that the Coxeter-Dynkin diagrams of D_3 and D_2 should be isomorphic with those of D_2 and A_1^2 respectively, as should the invariant degrees; and we readily confirm this as well. [On the Lie-group side, these correspond to the exceptional isogenies between the orthogonal group SO_6 and the linear group SL_4, and between SO_4 and $\mathrm{SL}_2^2 \sim \mathrm{SO}_3^2$, while SO_2 is abelian and thus has trivial

Lie-algebra structure.]

The root lattice D_4 has extra automorphisms: $\mathrm{Aut}(D_4)/W(D_4) \cong S_3$, permuting the nontrivial cosets in D_4^*/D_4. Explicitly, the normalized discrete Fourier transform on $(\mathbf{Z}/2\mathbf{Z})^2$,

$$\Phi_4 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \tag{7}$$

is an involution of D_4, and composing F with $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3, -x_4)$ (or any other single-coordinate negation) yields a 3-cycle. On the Lie-group side, the extra automorphisms of D_4 correspond to the "triality" automorphisms of the double cover Spin_8 of SO_8.

If n is even then there are two lattices other than $\mathbf{Z}^n$ that contain D_n and are contained in D_n^* with index 2. These two lattices are equivalent under $\mathrm{Aut}(D_n)$ (though not under $W(D_n)$), and we call either of them D_n^+; one is generated by D_n and $\frac{1}{2}\sum_{j=1}^n e_j$, the other by D_n and $-e_n + \frac{1}{2}\sum_{j=1}^n e_j$. The lattice D_n^+ is integral (and thus self-dual) if and only if $4|n$, and is even if and only if $8|n$. For $n = 4$ this lattice, being self-dual of rank at most 7, must be isomorphic with $\mathbf{Z}^4$; indeed it is taken to $\mathbf{Z}^4$ by one of the extra automorphisms of D_4, so for instance the columns of Φ_4 are an orthonormal basis for D_4^+. We have already encountered the even self-dual lattices D_8^+ and D_{16}^+, the former as E_8 and the latter as the unique even self-dual lattice in $\mathbf{R}^{16}$ other than E_8^2. In general, once $n > 8$ the lattice D_n^+ has no vectors of norm 1, and no roots other than those of D_n. For example, D_{12}^+ is the unique self-dual lattice in $\mathbf{R}^{12}$ with no vectors of norm 1, and D_{24}^+ is the Niemeier lattice with the maximal number of roots.

The theta series of D_n, D_n^*, and D_n^+ are easily obtained from $\theta_{\mathbf{Z}^n} = \theta_{\mathbf{Z}}^n$. Since D_n is the even sublattice of $\mathbf{Z}^n$, its theta series can be computed by removing from $\theta_{\mathbf{Z}}^n$ the terms with half-integral exponent; equivalently,

$$\theta_{D^n}(z) = \frac{1}{2}\Big(\theta_{\mathbf{Z}^n}(z) + \theta_{\mathbf{Z}^n}(z+1)\Big) = \frac{1}{2}\Big(\theta_{\mathbf{Z}}(z)^n + \theta_{\mathbf{Z}}(z+1)^n\Big). \tag{8}$$

[Likewise, for any integral lattice L the theta series of its even sublattice is $\frac{1}{2}\big(\theta_L(z) + \theta_L(z+1)\big)$.] The dual lattice D_n^* is the disjoint union of $\mathbf{Z}^n$ and the shadow of $\mathbf{Z}^n$, so

$$\theta_{D_n^*}(z) = \theta_{\mathbf{Z}^n}(z) + \psi_{\mathbf{Z}^n}(z) = \theta_{\mathbf{Z}}(z)^n + \psi_{\mathbf{Z}}(z)^n. \tag{9}$$

Finally, D_n^* is the disjoint union of D_n with half of the vectors in the shadow of $\mathbf{Z}^n$, so

$$\theta_{D_n^+}(z) = \frac{1}{2}\Big(\theta_{\mathbf{Z}}(z)^n + \theta_{\mathbf{Z}}(z+1)^n + \psi_{\mathbf{Z}}(z)^n\Big). \tag{10}$$

In particular, since $D_4^+ \cong \mathbf{Z}^4$ we deduce $\frac{1}{2}\big(\theta_{\mathbf{Z}}(z)^4 + \theta_{\mathbf{Z}}(z+1)^4 + \psi_{\mathbf{Z}}(z)^4\big) = \theta_{\mathbf{Z}}(z)^4$, which yields Jacobi's identity

$$\theta_{\mathbf{Z}}(z)^4 = \theta_{\mathbf{Z}}(z+1)^4 + \psi_{\mathbf{Z}}(z)^4. \tag{11}$$

The exceptional root lattices E_n, unlike the A_n and D_n lattices, cannot be found in any $\mathbf{Z}^N$. (It is enough to check this for $n = 6$, because E_6 is contained in E_7 and E_8. The E_6 diagram contains A_5

and D_5, and we readily check that either of these embeds into $\mathbf{Z}^N$ uniquely up to automorphism, and neither embedding extends to E_6.) One can still obtain explicit rational coordinates using our identification of E_7 with A_7^+, and of E_8 with A_8^{+3} or D_8^+; for E_6 we can use the slice of A_7^+ orthogonal to a minimal dual vector such as $(3,3,-1,-1,-1,-1,-1,-1)/4$. Of course we can also use the Gram matrix $2I-A$ of a base.

The Weyl groups $W(E_n)$ are so large that they exhaust the available automorphisms of $E_n/2E_n$. (The orders $\#W(E_n)$ can be computed from our table as $\prod_{j=1}^n d_j$.) Thus $W(E_8)$ maps to $\mathrm{Aut}(E_8/2E_8)$ with kernel $\{\pm 1\}$; the image must respect the quadratic form $\tilde{v} \mapsto \frac{1}{2}\langle v,v\rangle \bmod 2$, and the relevant orthogonal group $\mathrm{O}_8^+(\mathbf{Z}/2\mathbf{Z})$ is just large enough to accommodate $W(E_8)/\{\pm 1\}$, so $W(E_8)$ is a double cover of $\mathrm{O}_8^+(\mathbf{Z}/2\mathbf{Z})$. (While $\mathrm{O}_8^+(\mathbf{Z}/2\mathbf{Z})$ is not a sporadic group, its double cover *is* exceptional; see the table at the top of page *xvi* of the ATLAS of Conway et al. For the Weyl groups themselves, see pages 26, 46, and 85.) For E_7, we can identify $W(E_7)/\{\pm 1\}$ with the discriminant $+1$ subgroup of $W(E_7)$, and embed it in $\mathrm{Aut}(E_7/2E_7^*)$; this $(\mathbf{Z}/2\mathbf{Z})$-vector space of dimension 6 does not have a canonical quadratic form, but does inherit an alternating form from E_7, and again the corresponding symplectic group $\mathrm{Sp}_6(\mathbf{Z}/2\mathbf{Z})$ is barely large enough to contain the Weyl group, so we identify $W(E_7)$ with $\{\pm 1\} \times \mathrm{Sp}_6(\mathbf{Z}/2\mathbf{Z})$. Finally $W(E_6)$, though smaller than $W(E_7)$ and $W(E_8)$, is even more remarkable because the 51840-element group $W(E_6)$ is isomorphic with linear groups both over $\mathbf{Z}/2\mathbf{Z}$ and over $\mathbf{Z}/3\mathbf{Z}$: reducing mod 2 gives SO_6^- via the action on $E_6/2E_6$, and reducing mod 3 gives SO_5 via the action on $E_6/3E_6^*$. (Likewise the isomorphisms between S_n ($n \leq 6$) and small linear groups can be seen in the action of S_n on A_{n-1}; for example, $S_6 = W(A_5)$ can be identified with both $\mathrm{Sp}_4(\mathbf{Z}/2\mathbf{Z})$ and an orthogonal group acting on the 4-dimensional space $A_5/3A_5^{+3}$.)

The lattices E_n and their duals figure in the geometry of del Pezzo surfaces of degree $9-n$; the best known example is the 27 lines on a smooth cubic surface, which correspond to the minimal vectors in one of the nontrivial cosets of E_6 in E_6^*. (Each vector has norm $4/3$; two of them have inner product $-2/3$ if the corresponding lines intersect, and $+1/3$ if they do not. This and the E_7 picture can be recovered from intersection numbers on the del Pezzo surface.) The E_7 lattice has 56 dual vectors of minimal norm $3/2$, in 28 pairs corresponding to the bitangents to a smooth quartic curve; the dual vectors themselves biject with preimages of the bitangents in the double cover of $\mathbf{P}^2$ branched on the quartic. The 240 roots of E_8 correspond to minimal sections of a rational elliptic surface. We can continue the series E_n to some $n < 6$ by considering del Pezzo surfaces of higher degree: "E_5" is D_5, with 16 dual vectors of norm $5/4$ in a D_5 coset corresponding to the lines on the smooth intersection of two quadrics in $\mathbf{P}^4$; for $n=4$ we get $10 = \binom{5}{2}$ lines forming a Petersen-graph configuration, with "E_4" identified with A_4; and "E_3" is the reducible root lattice $A_1 \oplus A_2$. These can be seen by setting $e_3 = 2, 1, 0$ in the Coxeter-Dynkin diagram, as we did for $D_3 = A_3$ ($e_3 = 1$) and $D_2 = A_1^2$ ($e_3 = 0$).