Math 272y: Rational Lattices and their Theta Functions

23 and 25 September 2019: Theta functions of self-dual lattices

Suppose L is a self-dual, positive-definite lattice of rank n. Last time we showed that the theta function

$$\theta_L(z) := \Theta_L(e^{2\pi i z}) = 1 + \sum_{\substack{k>0 \\ N_k(L)\neq 0}} N_k(L) e^{\pi i k z} \tag{1}$$

(with $z \in \mathcal{H} = \{x + iy \in \mathbf{C} \mid y > 0\}$) satisfies functional equations that express $\theta_L(Sz)$ and $\theta_L(T^2 z)$ as simple multiples of $\theta_L(z)$, where S and T are the fractional linear transformations

$$S : z \mapsto -1/z, \qquad T : z \mapsto z + 1$$

(so $T^2 z = z + 2$). Namely, we showed that

$$\theta_L(T^2 z) = \theta_L(z), \quad \theta_L(Sz) = (z/i)^{n/2} \theta_L(z), \tag{2}$$

where "$(z/i)^{n/2}$" is the nth power of the principal square root of z/i (the square root with positive real part). Moreover, if L is even then the first identity in (2) can be replaced by $\theta_L(Tz) = \theta_L(z)$.

We can iterate these functional equations, obtaining a formula for $\theta_L(f(z))$ as a multiple of $\theta_L(z)$ where f is any fractional linear transformation in the group of fractional linear transformations generated by S, T^2 (or S, T for L even). We shall see that this makes $\theta_L(f(z))$ a modular form of weight $n/2$, which in turn places very strong constraints on the counts $N_k(L)$. In each case we begin by determining the structure of that group of fractional linear transformations.

Recall that over any field K the group of fractional linear transformations $z \mapsto (az + b)/(cz + d)$ ($ad - bc \neq 0$) is isomorphic with the group $\mathrm{PGL}_2(K)$ of invertible 2×2 matrices modulo scalars: the coset $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) K^*$ acts by $z \mapsto (az + b)/(cz + d)$. [This can be seen by direct computation, and explained by considering a fractional linear transformation as a linear change of projective coordinates $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} az+b \\ cz+d \end{smallmatrix}\right)$.] Our maps S, T correspond to the integer matrices

$$\mathsf{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \mathsf{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

of determinant 1; and $\mathsf{T}^2 = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$. Thus S and T generate some subgroup $\langle S, T \rangle$ of the "modular group"

$$\Gamma := \mathrm{SL}_2(\mathbf{Z})/\{\pm 1\} = \mathrm{PSL}_2(\mathbf{Z}),$$

acting on $\mathcal{H}$ by projective linear transformations; and S and T^2 generate some subgroup of $\langle S, T \rangle$.

Consider first the case of L even, for which θ_L transforms under all of $\langle S, T \rangle$. It is well-known[1] that in fact $\langle S, T \rangle$ is all of Γ. Serre gives one elegant argument at the start of Chap. VII of *A*

[1]The result that $\langle S, T \rangle = \Gamma$ can be proved in several ways. Perhaps the most elementary approach begins by observing that $ad - bc = 1$ implies $\gcd(a, b) = 1$ and recalling that conversely $\gcd(a, b) = 1$ implies that $ad - bc = 1$ has integer solutions (c, d) that can be computed using the extended Euclidean algorithm. This computation encodes a representation of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ as a word in $\langle \mathsf{S}, \mathsf{T} \rangle$. For example, given one solution (c, d) of $ad - bc = 1$, the complete solution is $\{(c + ma, d + mb) \mid m \in \mathbf{Z}\}$, and $\left(\begin{smallmatrix} a & b+ma \\ c & d+mc \end{smallmatrix}\right) = \mathsf{T}^m \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Course in Arithmetic, deriving this fact together with the fundamental domain $\mathcal{F} = \{z \in \mathcal{H} : |z| \geq 1, |\mathrm{Re}\, z| \leq 1/2\}$ of the action of Γ on $\mathcal{H}$:

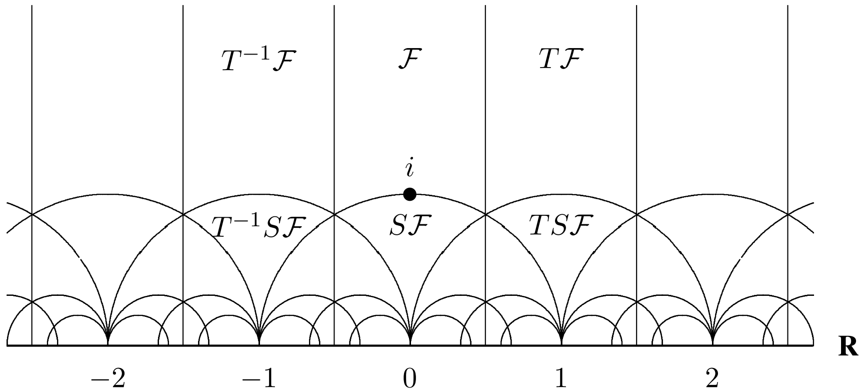

Figure 1: The fundamental domain $\mathcal{F}$ for Γ and some of its nearby images

Serre's approach is intimately related with lattice basis reduction in the Euclidean plane (identified with $\mathbf{C}$ in the usual way),[2] though he does not make this connection explicit.

We next review the connection between the structure of the group $\Gamma = \langle S, T \rangle$ with the hyperbolic geometry of the tiling of $\mathcal{H}$ by Γ-images of $\mathcal{F}$. Geometrically, $\mathcal{F}$ is an ideal isosceles triangle in the hyperbolic plane $\mathcal{H}$, with angles $\pi/3, \pi/3, 0$ at vertices $e^{\pi i/3}, e^{2\pi i/3}, \infty$. Our group Γ acts on $\mathcal{H}$ by conformal isometries. The positive imaginary axis bisects $\mathcal{F}$ into two right ideal triangles, each with angles $\pi/2, \pi/3, 0$ and thus congruent. Choose one of these halves, say the one with vertices $i, e^{2\pi i/3}, \infty$, and call it $\mathcal{T}$. The other half is then the image of $\mathcal{T}$ under the (anti-conformal) reflection $z \mapsto -\bar{z}$ in the imaginary axis. This reflection extends Γ to a group, call it $\overline{\Gamma}$, of hyperbolic isometries generated by reflections in the sides of $\mathcal{T}$. This identifies Γ with the conformal subgroup of $\overline{\Gamma}$, which is the *hyperbolic triangle group* of indices $2, 3, \infty$. We shall soon see that the group generated by S and T^2 (which acts on theta functions of odd self-dual lattices) is also a hyperbolic triangle group; we shall later encounter several other such groups as transformation groups for theta functions. We thus insert here a description of the general features of such groups.

Fix positive integers e_1, e_2, e_3 such that[3] $1/e_1 + 1/e_2 + 1/e_3 < 1$. There is a hyperbolic triangle (unique up to hyperbolic isometry) $\mathcal{T}(e_1, e_2, e_3)$ with angles $\pi/e_1, \pi/e_2, \pi/e_3$. Reflections in the sides of the triangle generate a group of hyperbolic isometries for which $\mathcal{T}(e_1, e_2, e_3)$ is a fundamental domain. For example, $(e_1, e_2, e_3) = (2, 4, 6)$ gives the hyperbolic tiling explored by Escher in his "Circle Limit" prints. The orientation-preserving subgroup is called the hyperbolic triangle

[2]…which in turn combines elements of the Euclidean algorithm (see the previous footnote) with Gram-Schmidt orthogonalization.

[3]If $1/e_1 + 1/e_2 + 1/e_3 = 1$ then the triangle is Euclidean, with (e_1, e_2, e_3) either $(3, 3, 3)$ (equilateral) or a permutation of $(2, 4, 4)$ (isosceles right triangle) or $(2, 3, 6)$ (a 30-60-90 triangle); allowing an ideal vertex we have also permutations of $(2, 2, \infty)$ (a half-strip). If $1/e_1 + 1/e_2 + 1/e_3 > 1$ then we have a spherical triangle; here (e_1, e_2, e_3) is up to permutation either $(2, 2, e)$, giving rise to a dihedral group, or one of the exceptional cases $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$, giving rise to the tetrahedral, octahedral, and icosahedral groups respectively.

group of indices (e_1, e_2, e_3); it is generated by rotations through angles $2\pi/e_1, 2\pi/e_2, 2\pi/e_3$ about the vertices of $\mathcal{T}(e_1, e_2, e_3)$. Call the rotations r_1, r_2, r_3. Then it is known that the hyperbolic triangle group is generated by r_1, r_2, r_3 subject only to $r_1^{e_1} = r_2^{e_2} = r_3^{e_3} = 1$ and the one additional relation $r_1 r_2 r_3 = 1$; equivalently, the group is

$$\langle r_1, r_2 \mid r_1^{e_1} = r_2^{e_2} = (r_1 r_2)^{e_3} = 1 \rangle.$$

Ideal vertices (cusps) are allowed: such a vertex has a zero angle and an index of ∞, and the "rotation" about such a vertex is parabolic element of $\mathrm{Aut}^+(\mathcal{H})$ fixing the cusp. In that case the relation $r_j^{e_j} = 1$ disappears; in the extreme case of $(e_1, e_2, e_3) = (\infty, \infty, \infty)$ the only relation is $e_1 e_2 e_3 = 1$ and we get a free group on two generators. It is also known that the triangle group has no elliptic or parabolic elements other than the conjugates of r_1, r_2, r_3.

In our setting, $e_1, e_2, e_3 = 2, 3, \infty$ in some order, and S and T are the generators that fix i and the cusp $i\infty$ respectively. Thus ST must fix the third vertex, and indeed if $z = e^{2\pi i/3}$ then

$$ST(z) = S(z+1) = -1/(z+1) = z$$

because $z^2 + z + 1 = 0$. We thus expect that $(ST)^3 = 1$, and can verify this directly either by computing $(ST)^3 z$ or by checking that $(\mathsf{ST})^3 = \binom{0\ -1}{1\ \ \ 1}^3 = -\mathbf{1}_2$. In fact $\langle S, T \mid S^2 = (ST)^3 = 1 \rangle$ is a presentation of Γ (a special case of the two-generator presentation of any triangle group), though we shall have little if any need for this fact.

We do, however, make good use of the identity $(ST)^3 = 1$. Suppose L is an even self-dual lattice. Then $\theta_L(z)$ transforms under $z \mapsto Tz = z + 1$ and $z \mapsto Sz = -1/z$. We noted already that iterating the functional equation

$$\theta_L(Sz) = (z/i)^{n/2} \theta_L(z)$$

gives $\theta_L(z) = \theta_L(S^2 z)$, which we knew already because $S^2 z = z$ for all z, and thus gives us only a "sanity check" on the functional equation. On the other hand, the identity $(ST)^3 z = z$ does give us something new: a proof that $8|n$. Indeed

$$\theta_L(ST(z)) = (T(z)/i)^{n/2} \theta_L(T(z)) = (T(z)/i)^{n/2} \theta_L(z)$$

for all $z \in \mathcal{H}$, so

$$\theta_L(z) = \theta_L((ST)^3 z) = \left((T(z)/i)^{1/2} \, (TST(z)/i)^{1/2} \, (TSTST(z)/i)^{1/2} \right)^n \theta_L(z). \qquad (3)$$

Now $\theta_L(z)$ is not identically zero (because $\lim_{y \to \infty} \theta_L(iy) = 1$), so the factor

$$\left((T(z)/i)^{1/2} \, (TST(z)/i)^{1/2} \, (TSTST(z)/i)^{1/2} \right)^n$$

in (3) must equal 1. On the other hand, we calculate $TST(z) = z/(z+1)$ and $TSTST(z) = S^{-1}(z) = S(z) = -1/z$, so

$$\frac{T(z)}{i} \, \frac{TST(z)}{i} \, \frac{TSTST(z)}{i} = i^{-3} (z+1) \Big(\frac{z}{z+1} \Big) \Big(\frac{-1}{z} \Big) = -1/i^3 = -i.$$

Hence $(T(z)/i)^{1/2}\,(TST(z)/i)^{1/2}\,(TSTST(z)/i)^{1/2}$ is a square root of $-i$ (which one?), and its nth power equals 1 if and only if $8|n$. This completes the proof that if a positive-definite lattice of rank n is even and self-dual then $8|n$.

Now the functional equation relating $\theta_L(z)$ with $\theta_L(Sz)$ simplifies to $\theta_L(Sz) = z^{n/2}\theta_L(z)$. Combining this with the translation invariance $\theta_L(Tz) = \theta_L(z)$, we deduce that for any $\binom{a\ b}{c\ d} \in \Gamma = \langle S, T\rangle$ the theta function satisfies

$$\theta_L\left(\frac{az+b}{cz+d}\right) = (cz+d)^{n/2}\theta_L(z). \tag{4}$$

Since θ_L is holomorphic on $\mathcal{H}$, and $\theta_L(z)$ remains bounded as $\mathrm{Im}(z) \to \infty$, this makes $\theta_L(z)$ a modular form of weight $n/2$ for Γ.

While the weight of θ_L is a multiple of 4, the transformation $(cz+d)^k\phi(z) = \phi(\gamma(z))$ is well-defined for all even k, because $(cz+d)^k$ is the same for either choice of sign in $\gamma = \pm\binom{a\ b}{c\ d}$. For any even integer $k \geq 0$, we say ϕ is a *modular form of weight k for* Γ if ϕ is a holomorphic function on $\mathcal{H}$ that satisfies $\phi(\gamma(z)) = (cz+d)^k\phi(z)$ for all $\gamma \in \Gamma$ and $\phi(z)$ remains bounded as $z \to i\infty$. We shall later need such forms also for $k \equiv 2 \bmod 4$, so we consider them together. For each k, the modular forms of weight k form a vector space. If ϕ_1, ϕ_2 are modular forms of weight k_1, k_2 then their product $\phi_1\phi_2$ is a modular form of weight $k_1 + k_2$. Moreover, a constant function is a modular form of weight zero, and there are no nonconstant weight-zero forms by a standard application of the maximum principle.[4] Thus we can package the modular forms of all weights into a graded algebra, which we call

$$\mathbf{M}(\Gamma) := \bigoplus_{\substack{k\geq 0\\ k \text{ even}}} M_k(\Gamma). \tag{5}$$

This algebra turns out to be freely generated by the normalized Eisentein series[5]

$$\mathsf{E}_4 \;=\; 1 + 240\sum_{m=1}^{\infty}\frac{m^3q^m}{1-q^m} = 1 + 240q + 2160q^2 + 6720q^3 + \cdots, \tag{6}$$

$$\mathsf{E}_6 \;=\; 1 - 504\sum_{m=1}^{\infty}\frac{m^5q^m}{1-q^m} = 1 - 504q - 16632q^2 - 122976q^3 - \cdots, \tag{7}$$

of weights 4 and 6 respectively. In general, for each even $k > 2$ we can construct a modular form E_k of weight k, with $\lim_{y\to\infty}\mathsf{E}_k(iy) = 1$, as the normalized Eisenstein series given by the

[4]Since ϕ is invariant under T, it descends to a holomorphic function of $q = e^{2\pi i z}$ on $0 \leq q \leq 1$; since this function is bounded near $q = 0$, the singularity at the origin is removable, and ϕ extends to a continuous function on a compact set, so $|\phi(\cdot)|$ has a maximum somewhere. But then it is constant near that maximum, whether or not it occurs at $q = 0$. Since $\mathcal{F}$ is connected, this makes ϕ a constant function.

[5]Warning: there is an unfortunate but unavoidable notational collision here between the lattice E_8 and the Eisenstein series E_k. Even worse, the usual indexing of Eisenstein series makes θ_{E_8} equal E_4, not E_8 which is the theta function of each of the Type II lattices in $\mathbf{R}^{16}$. We always use a sans serif E for the Eisenstein series — which also suggests the mnemonic that this E, being thin, has half the weight.

absolutely convergent sums

$$\mathsf{E}_k = \frac{1}{2\zeta(k)} \sum\sum_{\substack{c,d\in\mathbf{Z}^2\\(c,d)\neq(0,0)}} \frac{1}{(cz+d)^k} = 1 - \frac{2k}{B_k}\sum_{m=1}^{\infty}\frac{m^{k-1}q^m}{1-q^m} = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n. \tag{8}$$

Here B_k is the k-th Bernoulli number, so the coefficients $-2k/B_k$ for $k = 4, 6, 8, 10, 12, 14, 16, \ldots$ are $240, -504, 480, -264, 65520/691, -24, 16320/3617, \ldots$, and $\sigma_{k-1}(n) = \sum_{m|n} m^{k-1}$. The sum over (c, d) is readily seen to be modular. We refer to the classic exposition in Serre's *A Course in Arithmetic* for a proof of the remaining equality in (8) and of the next theorem (his Theorem 4 in Chapter VII):

Theorem. *The algebra $\mathbf{M}(\Gamma)$ is freely generated over $\mathbf{C}$ by the modular forms E_4 and E_6. In other words, each $M_n(\Gamma)$ $(n = 0, 2, 4, \ldots)$ has basis*

$$\{\mathsf{E}_4^a\mathsf{E}_6^b : a, b \geq 0, 4a + 6b = n\}. \tag{9}$$

Hence $\dim M_n(\Gamma)$ is the number of possible (a, b) in (9), which is $\lfloor n/24\rfloor$ if $n \equiv 2 \bmod 12$ and $1 + \lfloor n/24\rfloor$ for other even $n \geq 0$:

n	$\dim M_n$	n	$\dim M_n$	n	$\dim M_n$	$\cdots$
0	1	12	2	24	3	$\cdots$
2	0	14	1	26	2	$\cdots$
4	1	16	2	28	3	$\cdots$
6	1	18	2	30	3	$\cdots$
8	1	20	2	32	3	$\cdots$
10	1	22	2	34	3	$\cdots$

(10)

So for example $\mathsf{E}_8 = \mathsf{E}_4^2$, $\mathsf{E}_{10} = \mathsf{E}_4\mathsf{E}_6$, and $\mathsf{E}_{14} = \mathsf{E}_4^2\mathsf{E}_6$; these identities encode surprising convolution identities involving the divisor functions $\sigma_3, \sigma_5, \sigma_7, \sigma_9, \sigma_{13}$. ("Elementary" but tricky proofs are known.)

The structure of $\mathbf{M}(\Gamma)$ has the following key consequence for theta functions:

Corollary. *Let L be a positive-definite self-dual lattice of rank n. If L is even then*

$$\theta_L = \mathsf{E}_4^{n/8} + \sum_{m=1}^{\lfloor n/24\rfloor} c_m \mathsf{E}_4^{n-24m}\Delta^m \tag{11}$$

for some constants c_m $(m = 1, 2, \ldots, \lfloor n/8\rfloor)$, where

$$\Delta := \frac{\mathsf{E}_4^3 - \mathsf{E}_6^2}{12^3} = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - 16744q^7 \cdots \tag{12}$$

Proof: Because $8|n$, any element of $M_{n/2}(\Gamma)$ is a polynomial in E_4 and E_6^2. We may thus write $\theta_L = \sum_{m=0}^{\lfloor n/24 \rfloor} c_m \theta_{\mathbf{Z}}^{n-8m} \Delta^m$ for some c_m ($m = 0, 1, 2, \ldots, \lfloor n/24 \rfloor$), and evaluate at $q = 0$ to obtain $c_0 = N_0(L) = 1$ as desired.

It follows that the coefficients $N_{2k}(L)$ of θ_L for $k = 1, 2, \ldots, \lfloor n/24 \rfloor$ determine θ_L, because we can use them to calculate the c_m iteratively.

Note that when $n < 24$ there are no undetermined coefficients, so $\theta_L = \mathsf{E}_4^{n/8} = 1 + 30nq + O(q^2)$. Once we classify "root lattices" (integral positive-definite lattices generated by their vectors of norm 2), we will see almost immediately that L is isomorphic with E_8 if $n = 8$, and with either $E_8 \oplus E_8$ or D_{16}^+ (Serre's "E_{16}") if $n = 16$; that is, we'll obtain the classification of even unimodular lattices in $\mathbf{R}^n$ for $n < 24$.

In fact for $n = 8$ we can already obtain this result by including one more term of the theta function,[6] $\theta_L = \mathsf{E}_4 = 1 + 240q + 2160q^2 + O(q^3)$. There are barely enough cosets of L in $2L$ to accommodate 0 and the $240 + 2160$ vectors of norm 2 or 4. Indeed suppose v, v' are vectors of norm at most 4 such that $v' \neq \pm v$ butt $v \equiv v' \bmod 2L$. Then both $v + v'$ and $v - v'$ are in $2L$, and being nonzero vectors must have $\langle v + v', v + v' \rangle \geq 8$ and $\langle v - v', v - v' \rangle \geq 8$. But then

$$16 = 8 + 8 \leq \langle v + v', v + v' \rangle + \langle v - v', v - v' \rangle = 2(\langle v, v \rangle + \langle v', v' \rangle) \leq 2(4 + 4) = 16.$$

Hence equality holds throughout, so $\langle v, v \rangle = \langle v', v' \rangle = 4$ — and then $\langle v, v' \rangle = 0$ because $\langle v + v', v + v' \rangle = 8$. This means that any coset of L in $2L$ that contains a pair $\pm v$ of norm-2 vectors contains no other vectors of norm at most 4, and any coset of minimal norm 4 contains at most 8 pairs $\pm v_i$ of norm-4 vectors. But this requires $240/2 + 2160/16 = 120 + 135 = 255$ nonzero cosets of L in $2L$, which exactly matches the total number $2^8 - 1$ of such cosets. So again equality holds throughout. In particular, for each of the 135 cosets containing a vector of norm 4 we find an orthogonal frame $\pm v_1, \ldots, \pm v_8$ of such vectors. Since these are all congruent mod $2L$, each vector $(v_i - v_j)/2$ is also in L; and this gives a sublattice of L congruent to D_8, the even sublattice of $\mathbf{Z}^8$. But then $D_8 \subset L = L^* \subset D_8^*$. There are only three lattices properly contained between D_8 and D_8^*; one is $\mathbf{Z}^8$ itself, and the others are obtained by augmenting D_8 by its translate by $(1, 1, 1, 1, 1, 1, 1, 1)/2$ or $(1, 1, 1, 1, 1, 1, 1, -1)/2$. All three are self-dual, but only the latter two are even, and they are isomorphic to each other by flipping the sign of any coordinate (which preserves the index-2 sublattice D_8). This completes the proof, and also lets us count automorphisms of E_8: the 135 cosets of norm-4 vectors are all equivalent, and each such coset's stabilizer is half of $\mathrm{Aut}(D_8)$, so

$$\#(\mathrm{Aut}\, E_8) = \frac{135}{2} \#(\mathrm{Aut}\, D_8) = 135 \cdot 2^7 8! = 696729600.$$

Moreover, it follows that $\mathrm{Aut}\, E_8$ acts transitively on the vectors of norm 4, and (with a bit more work) on norm-2 vectors, indeed even on ordered orthogonal pairs of norm-2 vectors. We shall see that $\mathrm{Aut}\, E_8$ acts transitively on many more such configurations of short vectors.

[6]This proof is modeled after (but quicker and easier than) Conway's short characterization of the Leech lattice and enumeration of its automorphisms, published in *Inventiones Math.* in 1969 and reproduced in SPLAG as Chapter 12. We shall encounter several further lattices that allow for similar characterizations and counts. For E_8 there are many other proofs, some of which we shall give later in this course.

What then of the subgroup of Γ generated by S and T^2? Call this subgroup Γ_+. Since T^2 is congruent to the identity mod 2, while $S^2 = 1$, any $\pm\binom{a\,b}{c\,d} \in \Gamma_+$ must be congruent to either the identity or $\binom{0\,1}{1\,0} \bmod 2$. We claim that this necessary condition is also sufficient; that is, that Γ_+ is the preimage in Γ of $\{\binom{1\,0}{0\,1}, \binom{0\,1}{1\,0}\}$ under the reduction-mod-2 map $\mathrm{PSL}_2(\mathbf{Z}) \to \mathrm{SL}_2(\mathbf{Z}/2\mathbf{Z})$, and thus that the index $[\Gamma : \Gamma_+]$ is $\#(\mathrm{SL}_2(\mathbf{Z}/2\mathbf{Z}))/2 = 3$. One way to see this is to check that Γ_+ is itself a hyperbolic triangle group, this time with indices $2, \infty, \infty$. Indeed S is an involution, T^2 is parabolic, and $ST^2 : z \mapsto -1/(z+2)$ is also parabolic, fixing the cusp $z = -1$. Thus $[\Gamma : \Gamma_+]$ is the ratio of the areas of the corresponding hyperbolic triangles, which is $(1 - \frac{1}{2})/(1 - \frac{1}{2} - \frac{1}{3}) = \frac{1}{2}/\frac{1}{6} = 3$; since we have already found an index-3 subgroup of Γ that contains Γ_+, that must be the same as Γ_+ and we're done. (Alternatively, prove that $[\Gamma : \Gamma_+] \le 3$ by checking group-theoretically that each element of $\langle S, T \mid S^2 = (ST)^3 = 1\rangle$ is either in $\langle S, T^2\rangle$ or in the $(\langle S, T^2\rangle)$-coset of T or TS, and conclude the proof as before.) as with Γ, we can construct a fundamental domain $\mathcal{F}_+$ for Γ_+, consisting of two copies of the associated hyperbolic triangle, which comprise a fully ideal triangle with vertices at $-1, 1, i\infty$; note that -1 and 1 are equivalent under Γ_+ (as the vertices $e^{\pi i/3}$ and $e^{2\pi i/3}$ of $\mathcal{F}$ are equivalent under Γ). As expected, $\mathcal{F}_+$ can be dissected into hyperbolic triangles that form three Γ-images of $\mathcal{F}$.

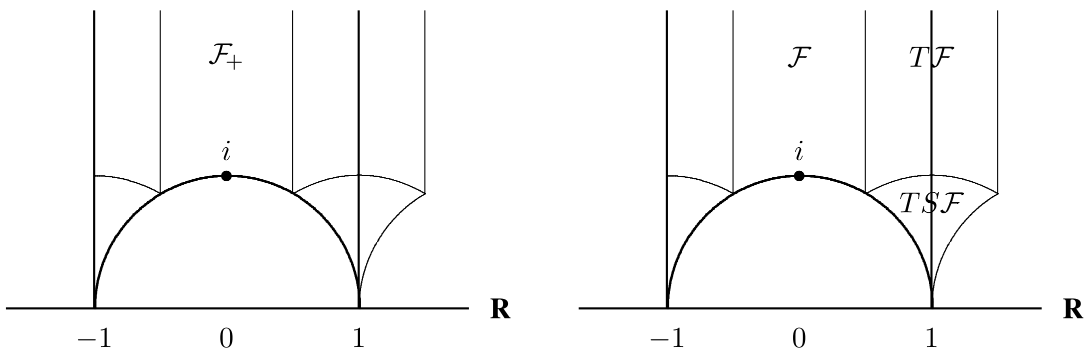

Figure 2: The fundamental domain $\mathcal{F}_+$ dissected into three images of $\mathcal{F}$

So, if L is self-dual then we can iterate our functional equations (2) to find, for each $\gamma = \pm\binom{a\,b}{c\,d} \in \langle S, T^2\rangle = \Gamma_+$, an identity

$$\theta_L(\gamma(z)) = \epsilon_{c,d}^n (cz+d)^{n/2} \theta_L(z) \tag{13}$$

for some $\epsilon_{c,d} \in \mathbf{C}^*$ with $\epsilon_{c,d}^8 = 1$.[7] We shall use this as before to prove that θ_L is in the algebra freely generated by θ_Z and E_4. But there are a few new features here, due to the second cusp and the factors $\epsilon_{c,d}^n$, that we must address before we can mimic the analysis of $\mathbf{M}(\Gamma)$ in Serre. Along the way we shall encounter another important identity relating the theta functions of L and its shadow.

A holomorphic function $\phi : \mathcal{H} \to \mathbf{C}$ satisfying $\phi(\gamma(z)) = \epsilon_{c,d}^n (cz+d)^{n/2} \phi(z)$ for all $\gamma = \pm\binom{a\,b}{c\,d} \in \Gamma_+$ is a weakly modular form of weight $n/2$ for Γ_+ (more fully, for Γ_+ and the $\epsilon_{c,d}^n$; these

[7]We write $\epsilon_{c,d}$ rather than ϵ_γ because if some γ' has the same (c,d) as γ then $\gamma' = T^{2k}\gamma$ for some $k \in \mathbf{Z}$, so $\theta_L(\gamma(z)) = \theta_L(\gamma'(z))$, whence γ and Γ' have the same ϵ; also, γ determines (c,d) only up to sign, and changing (c,d) to $(-c,-d)$ multiplies $\epsilon_{c,d}$ by $\pm i$.

factors are said[8] to form a "multiplier system of weight $n/2$" — a multiplier system being a system of factors in an identity such as (13) that is consistent with $\theta_L(\gamma_1(\gamma_2(z)) = \theta_L((\gamma_1\gamma_2)(z))$ for all choices of γ_1, γ_2). To remove the adverb "weakly", we must check that $\theta_L(z)$ does not grow too quickly as z approaches a cusp. All cusps are equivalent under Γ, but not under Γ_+; so we shall determine the action on θ_L of our representatives T, TS of the nontrivial cosets of Γ_+ in Γ.

The action of $T : z \mapsto z + 1$ is easy: this map takes $e^{\pi iz}$ to $-e^{\pi iz}$, so for any integral lattice L we have simply

$$\theta_L(T(z)) = \theta_L(z+1) = 1 + \sum_{k=1}^{\infty} (-1)^k N_k(L) e^{\pi ikz}.$$

The formula for $\theta_L(TS(z))$ can be expressed in terms of the theta series for the shadow of L. Recall that the "shadow" is the lattice translate consisting of all vectors $c/2$ for c in the characteristic coset of $2L$ in L; we denote the shadow by $s(L)$. Let us define

$$\Psi_L(q) \ := \ \sum_{v \in s(L)} q^{\langle v,v\rangle/2} = 1 + \sum_{k=1}^{\infty} N_{2k}(s(L)) q^k, \tag{14}$$

$$\psi_L(z) \ := \ \Psi_L(e^{2\pi iz}) = \sum_{\substack{k>0 \\ N_k(s(L))\neq 0}} N_k(s(L)) e^{\pi ikz}, \tag{15}$$

where

$$N_k(s(L)) = \#\{v \in s(L) \mid \langle v,v\rangle = k\}. \tag{16}$$

Then we have:

Proposition. *For any self-dual lattice L in $\mathbf{R}^n$ we have*

$$\theta_L(TS(z)) = (z/i)^{n/2} \psi_L(z) \tag{17}$$

for all $z \in \mathcal{H}$.

It will follow that $|z|^{-n/2}\theta_L(TS(z))$ remains bounded as $z \to i\infty$, which is what we need to check at the cusp $1 = TS(\infty)$ to verify that θ_L is modular (not just weakly modular) for Γ_+.

Proof of (17): Let c be a characteristic vector. Then

$$\theta_L(T(z)) = \theta_L(z+1) = \sum_{v\in L} (-1)^{\langle v,v\rangle} e^{\pi i\langle v,v\rangle z} = \sum_{v\in L} e^{\pi i(\langle v,v\rangle z + \langle v,c\rangle)} \tag{18}$$

because $(-1)^c = e^{\pi ic}$ for any integer c. We now apply Poisson summation to the sum. The Fourier transform of $\exp \pi i(\langle x,x\rangle z + \langle x,c\rangle)$ is the integral over $x \in \mathbf{R}^n$ of $\exp(\pi i(\langle x,x\rangle z + \langle x, c+2y\rangle))$, which is to say the value at $y + \frac{c}{2}$ of the Fourier transform of $\exp(\pi i\langle x,x\rangle z)$, which we already know is $(z/i)^{-n/2} \exp(-\pi i\langle x,x\rangle/z)$. Poisson summation then gives

$$\theta_L(T(z)) = (z/i)^{-n/2} \sum_{v\in L} e^{-\pi i\langle v+\frac{c}{2}, v+\frac{c}{2}\rangle/z} = (z/i)^{-n/2} \sum_{v\in s(L)} e^{-\pi i\langle v,v\rangle/z} \tag{19}$$

[8] See e.g. Iwaniec's *Topics in Classical Automorphic Forms*, 2.6.

which is $(z/i)^{-n/2}\psi_L(S(z))$; replacing z by Sz we recover (17), Q.E.D.

Corollary. *If L is a positive-definite self-dual lattice of rank n then every characteristic vector c has $\langle c,c\rangle \equiv n \bmod 8$.*

Proof: We have seen that all characteristic vectors have the same norm mod 8; denote this common residue mod 8 by s. Then $\psi_L(t+1) = e^{\pi i s(L)/4}\psi_L(t)$. We claim that $s=n$, or equivalently that

$$\psi_L(t+1) = e^{\pi i n/4}\psi_L(t). \tag{20}$$

Using (17), together with $S^2 = (ST)^3 = 1$ and $\theta_L(T^2 z) = \theta_L(z)$, we calculate

$$\left(\frac{t+1}{i}\right)^{n/2}\psi_L(t+1) = \theta_L(TST(t)) = \theta_L(ST^{-1}S(t))$$

$$= (T^{-1}S(t)/i)^{n/2}\theta_L(T^{-1}S(t)) = \left(\frac{i(t+1)}{t}\right)^{n/2}\theta_L(TS(t))$$

(in which we used $S^2 = (ST)^3 = \mathbf{1}$ and the invariance of θ_L under T^2, and again wrote $n/2$ power to mean nth power of principal square root). Comparing with (17) yields the desired identity (20), Q.E.D.

Now to describe the $\frac{1}{2}\mathbf{Z}$-graded algebra

$$\mathbf{M}(\Gamma_+) := \bigoplus_{\substack{k\ge 0 \\ 2k\in\mathbf{Z}}} M_k(\Gamma_+). \tag{21}$$

of modular forms for Γ_+. Recall (e.g. Theorem 3 in Serre, Chap. VII) that any nonzero modular form of weight k for Γ has $k/12$ zeros in $\mathcal{F}$, counted with appropriate multiplicity (including zeros at the cusp, and zeros at i and $e^{2\pi i/3}$ counted with half the usual multiplicity). A similar result and proof works for Γ_+ and $\mathcal{F}_+$, but with a total of $k/4$ — as it must be because if we start with a form in $M_k(\Gamma)$ then we must multiply $k/12$ by the index $[\Gamma : \Gamma_+]$ to get the total multiplicity in $\mathcal{F}_+$. A new feature is that due to the "multiplier system" the multiplicity of a zero at the cusp ± 1 is $\equiv n/8 \bmod 1$; we have in effect seen this in our proof of $\langle c,c\rangle \equiv n \bmod 8$. At $i\infty$, we assign multiplicity m to the zero of $q^{m/2} + O(q^{(m+1)/2})$. In the determination of $\mathbf{M}(\Gamma)$ it was crucial that the weight-12 form Δ had a zero at $i\infty$ and nowhere else in $\mathcal{F}$. For Γ_+, we can use instead the weight-4 form we call

$$\Delta_+ = \frac{1}{16}(\theta_{\mathbf{Z}}^8 - \mathsf{E}_4) = q^{1/2} - 8q + 28q^{3/2} - 64q^2 + 126q^{5/2} - 224q^3 + 344q^{7/2} - 512q^4 \cdots .$$

Since Δ_+ vanishes at $i\infty$, it has no other zeros; thus for each half-integer k we see that multiplication by Δ_+ is an isomorphism from $M_{k-4}(\Gamma_+)$ to the subspace of $M_k(\Gamma_+)$ consisting of forms vanishing at $i\infty$. For $k \ge 0$ this subspace has codimension 1 because it is the kernel of a linear function and does not contain $\theta_{\mathbf{Z}}^{2k}$. We now have all the ingredients to prove, exactly as Serre does for $\mathbf{M}(\Gamma)$:

Theorem. *The algebra* $\mathbf{M}(\Gamma_+)$ *is freely generated over* $\mathbf{C}$ *by the modular forms* $\theta_{\mathbf{Z}}$ *and* E_4*. In other words, each* $M_{n/2}(\Gamma)$ $(n = 0, 1, 2, 3, \ldots)$ *has basis*

$$\{\theta_{\mathbf{Z}}^a \mathsf{E}_4^b : a, b \geq 0, a + 8b = n\}. \tag{22}$$

An easy consequence is the classification of positive-definite self-dual lattices of rank up to 7:

Proposition. *If* $n < 8$ *then every self-dual lattice in* $\mathbf{R}^n$ *is isomorphic with* $\mathbf{Z}^n$*.*

Proof: Here $\lfloor n/8 \rfloor = 0$, so $\theta_L = \theta_{\mathbf{Z}^n}$. Comparing coefficients, we deduce $N_k(L) = N_k(\mathbf{Z}^n)$ for all k. In particular, $N_1(L) = 2n$, because $N_1(\mathbf{Z}^n) = 2n$, as may be seen either directly or by expanding $\theta_{\mathbf{Z}}^n$ in powers of $q^{1/2}$. Thus L contains n pairs $\pm v_i$ $(1 \leq i \leq n)$ of vectors with $\langle v_i, v_i \rangle = 1$. For $i \neq j$ we then have $|\langle v_i, v_j \rangle| < 1$ by Cauchy-Schwarz; since L is integral, it follows that $\langle v_i, v_j \rangle = 0$. That is, the v_i are orthonormal. Therefore L contains their $\mathbf{Z}$-span, call it L_0, which is isomorphic with $\mathbf{Z}^n$. But then $L_0 \subseteq L = L^* \subseteq L_0^* = L_0$, so $L = L_0 \cong \mathbf{Z}^n$, Q.E.D.

For any n, the characteristic vectors in $\mathbf{Z}^n$ all have norm at least n (with equality for the 2^n vectors $\pm 1, \ldots, \pm 1)$), so $\psi_{\mathbf{Z}^n}(z) = O(|q|^{n/4})$ as $z \to i\infty$. Thus $\theta_{\mathbf{Z}^n}$ has a zero at the cusp $z = \pm 1$ that contributes $n/8$ to the total count of zeros with multiplicity. Again we deduce, as with Δ and Δ_+, that there are no other zeros. We conclude this chapter of the notes by using that observation about $\theta_{\mathbf{Z}^n}$ to prove that $\mathbf{Z}^n$ is the unique self-dual lattice in $\mathbf{R}^n$ whose characteristic vectors all have norm at least n. If L is any such lattice then $\psi_L(z) = O(|q|^{n/4})$ as $z \to i\infty$, so θ_L vanishes at the cusp $z = \pm 1$ to at least the same order as $\theta_{\mathbf{Z}^n}$. Hence $\theta_L / \theta_{\mathbf{Z}^n}$ is a modular form of weight 0, and is thus constant; as usual, it follows from $\Theta_L(0) = 1 = \Theta_{\mathbf{Z}^n}(0)$ that $\theta_L = \theta_{\mathbf{Z}^n}$. In particular, L has n pairs of vectors of norm 1. This implies $L \cong \mathbf{Z}^n$, using the same argument as in the previous paragraph. This answers a question that arose in the geometry of 4-manifolds. With some further work (including the use of root lattices), this technique even lets us classify all self-dual lattices in $\mathbf{R}^n$ with no characteristic vectors of norm less than $n - 8$.[9]

[9] See my papers "A characterization of the $\mathbf{Z}^n$ lattice" and "Lattices and codes with long shadows" in *Math. Research Letters* **2** (1995), 321–326 and 643–651 (arXiv: math.NT/9906019 and math.NT/9906086).