Math 272y: Rational Lattices and their Theta Functions

28 and 30 October 2019: The Niemeier lattices

We saw that the classification of even unimodular lattices in $\mathbf{R}^8$ and $\mathbf{R}^{16}$ is almost immediate from the classification of root lattices together with the result that the theta functions must be E_4 and $\mathsf{E}_8 = \mathsf{E}_4^2$ respectively. For rank 24 the classification is still tractable but much harder: the theta function has an undetermined coefficient (it is $\mathsf{E}_4^3 + c\Delta$ for some $c \in \mathbf{Z}$), and there are many possible root lattices, not all of rank 24. The even unimodular lattices in $\mathbf{R}^{24}$ were first classified by Niemeier in his doctoral thesis (1968, published five years later[1]), and are often called the *Niemeier lattices*. There are 24; one has no roots, and is the famous Leech lattice Λ_{24}, while each of the other 23 Niemeier lattices has enough roots to span a lattice $R(N)$ of finite index in N. In particular, Λ_{24} is the unique even self-dual lattice of rank 24 with no roots. The proof is long and computational (we shall say more about this at the end of this chapter of the notes), and yields confirmation of the result but not much understsanding. Conway soon used the theta function to give a much simpler proof[2] of the uniqueness of Λ_{24} and its count of automorphisms, as part of his study of $\mathrm{Aut}(\Lambda_{24})$ which led to three new sporadic simple groups Co_1, Co_2, Co_3 and much more. Venkov[3] then used weighted theta functions to simplify the rest of the proof; we follow his argument here.

Specifically, Venkov uses the modularity of $\theta_{N,P}$ for harmonic P of degree 2 to prove *a priori* that:

Theorem 1. *If N is a Niemeier lattice then either N has no roots or $R(N)$ has rank 24. Moreover, in the latter case the simple components of $R(N)$ all have the same Coxeter number.*

(Recall that the *Coxeter number* of a simple root lattice R is the integer $h(R) = N_2(R)/\mathrm{rank}\,R$; we have seen that $h(A_n) = n + 1$, $h(D_n) = 2n - 2$, and $h(E_n) = 12, 18, 30$ for $n = 6, 7, 8$.)

There are 23 root lattices R of rank 24 and constant Coxeter number. Given such a lattice, recovering N is a finite computation: we need an even self-dual lattice betwen R and R^* all of whose roots are already in R. In particular, the index $[N : R]$ must be $\mathrm{Vol}(\mathbf{R}^{24}/R) = \mathrm{disc}(R)^{1/2}$, so a necessary condition is that $\mathrm{disc}\,R$ be a square — and remarkably this is the case for each of the 23 allowed R! See the following table (which also includes the $h = 0$ case of a zero root lattice):

h	0	2	3	4	5	6	6	7	8	9	10	10
$R(N)$	(0)	A_1^{24}	A_2^{12}	A_3^8	A_4^6	$A_5^4 D_4$	D_4^6	A_6^4	$A_7^2 D_5^2$	A_8^3	$A_9^2 D_6$	D_6^4
$[N : R(N)]$	∞	2^{12}	3^6	256	125	72	64	49	32	27	20	16

h	12	12	13	14	16	18	18	22	25	30	30	46
$R(N)$	$A_{11}D_7E_6$	E_6^4	A_{12}^2	D_8^3	$A_{15}D_9$	$A_{17}E_7$	$D_{10}E_7^2$	D_{12}^2	A_{24}	$D_{16}E_8$	E_8^3	D_{24}
$[N : R(N)]$	12	9	13	8	8	6	4	4	5	2	1	2

A further "miracle" is that each of these R arises, and uniquely:

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[1]Hans-Volker Niemeier: Definite quadratische Formen der Dimension 24 und Diskriminante 1, *J. Number Theory* **5**, 142–178 (1973).

[2]See SPLAG, Chapter 12.

[3]B. B. Venkov, The classification of integral even unimodular 24-dimensional quadratic forms, *Proc. Steklov Inst. of Math.* **4**, 63–74 (1980); published two years earlier in Russian, and isomorphic with Chapter 18 of SPLAG.

Theorem 2. *For each of the $1 + 23$ root lattices R satisfying the conditions of Theorem 1, there is a unique Niemeier lattice with root lattice R.*

We prove Theorem 1, and then give some of the computations and combinatorics that go into Venkov's proof of Theorem 2; for the fuller details, see his paper.

Proof of Theorem 1: For any harmonic polynomial P of degree 2, the weighted theta function $\theta_{N,P}$ is a cusp form of weight 14 for $\Gamma(1)$, and thus vanishes identically. In particular (using the q^1 coefficient), the sum of $P(v)$ over all roots v is zero. If e, e' are any unit vectors (not necessarily orthogonal) we may take $P(x) = \langle x, e \rangle^2 - \langle x, e' \rangle^2$ to deduce that

$$c := \sum_{\substack{v \in L \\ \langle v, v \rangle = 2}} \langle v, e \rangle^2 \tag{1}$$

is independent of the choice of unit vector c. Then $c = 0$ if N has no roots and $c > 0$ otherwise. If N does have roots then then they must generate $\mathbf{R}^{24}$, else we would get a contradiction by taking e in the orthogonal complement of $R(N)$. Now let $e_1, \ldots, e_n$ be an orthogonal basis for a simple component R_0 of $R(N)$. Then

$$nc = \sum_{\substack{v \in L \\ \langle v, v \rangle = 2}} \left(\sum_{j=1}^{n} \langle v, e_j \rangle^2 \right), \tag{2}$$

and the inner sum is $\langle v, v \rangle = 2$ if $v \in R_0$ and zero otherwise. Hence that component has $nc/2$ roots and Coxeter number $h = c/2$, so indeed h is independent of the choice of R_0 and we are done. □

To prove Theorem 2 we consider each possible $R = R(N)$ in turn. Roughly speaking, the difficulty of the computation increases with the size of the discriminant group R^*/R, and thus decreases with h, as suggested by the following plot of the 23 pairs $(h, \frac{1}{2} \log_2 |R^*/R|) = (h, \log_2(\mathrm{Vol}(\mathbf{R}^{24}/R)))$:
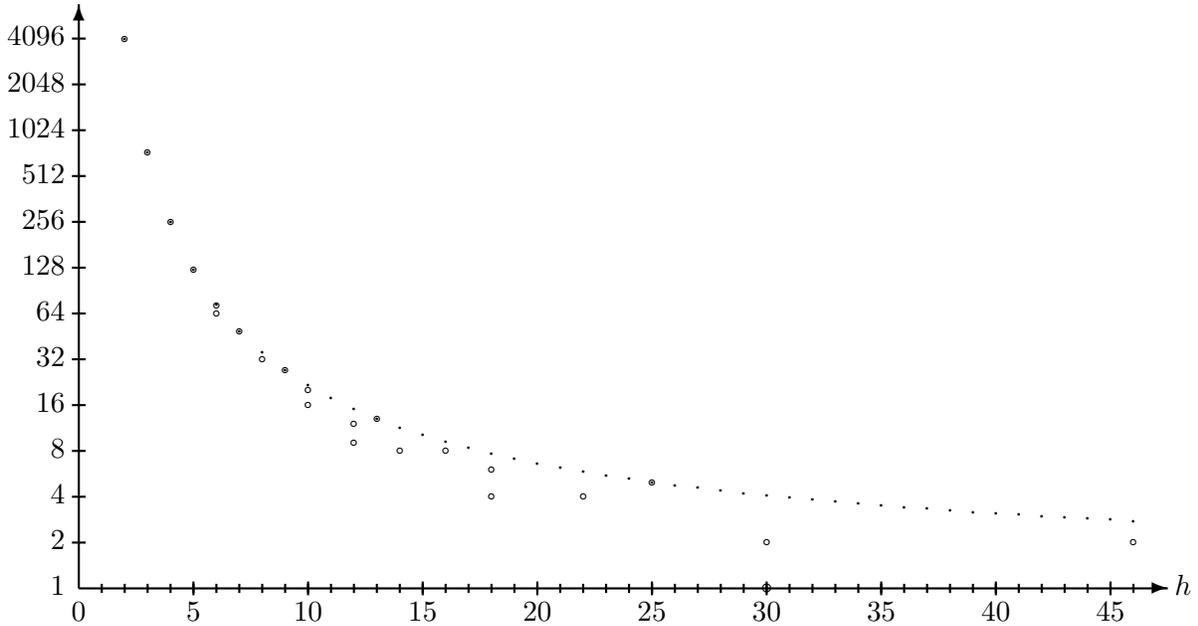
Coxeter numbers h, root-lattice covolumes of Niemeier lattices, and the upper bound $h^{12/(h-1)}$

The circles show $(h, \log_2 \operatorname{disc} R)$, and the dots show the upper bound $|\operatorname{disc} R| \leq h^{12/(h-1)}$, attained if and only if each component of R is A_{h-1}.

We next account for the easier 12 of the 24 possible $R(N)$, those with $h \geq 12$ ($\Longleftrightarrow \operatorname{disc} R \leq 13$). We shall then skip ahead to the cases $R(N) = A_2^{12}$ and A_1^{24} ($h = 3$ and $h = 2$) and the Leech lattice ($h = 0$); see Venkov's paper or other sources for the remaining nine cases with $4 \leq h \leq 10$.

$\underline{h = 46}$: Here $R(N) = D_{24}$, so $N = D_{24}^+$, spanned by D_{24} and the vector $\frac{1}{2} \sum_{j=1}^{24} e_j$ (or the other half-integral coset of D_{24} in D_{24}^*, which gives an isomorphic lattice).

$\underline{h = 30}$: If $R(N) = E_8^3$ then of course $N = E_8^3$. If $R(N) = D_{16} \oplus E_8$ then E_8 is a direct summand, so $N = D_{16}^+ \oplus E_8$.

$\underline{h = 25}$: We already noted $N = A_{24}^{+5}$, which is the unique lattice properly between A_{24} and A_{24}^*.

$\underline{h = 22}$: If $R(N) = D_{12}^2$ then $D_{12}^2 \subset N \subset (D_{12}^*)^2$, both containments having index 4. The nontrivial cosets of D_{12} in D_{12}^* have minimal norms $1, 3, 3$, with 1 arising for the coset consisting of $\mathbf{Z}^{12}$ vectors of odd norm. So we need the graph of a map from D_{12}^*/D_{12} to itself that does not take the integral coset to itself (else we would be back to D_{24}^+). There are four such maps, determined uniquely by the image and preimage of the integral coset, and all are equivalent under automorphisms of D_{12}^2.

$\underline{h = 18}$: For $R(N) = D_{10} \oplus E_7^2$, the minimal coset norms are $0, 1, \frac{5}{2}, \frac{5}{2}$ for D_{10} and $0, \frac{3}{2}$ for E_7. The even combinations are

$$0 + 0 + 0, \quad 1 + \frac{3}{2} + \frac{3}{2}, \quad \frac{5}{2} + \frac{3}{2} + 0, \quad \frac{5}{2} + 0 + \frac{3}{2},$$

and we must use one of each kind. There are two choices (depending on a bijection between the half-integral D_{10} cosets and the E_7 components), both equivalent under $\operatorname{Aut}(D_{10})$. For $R(N) = A_{17} \oplus E_7$, we can only use A_{17} cosets in $A_{17} + 3A_{17}^*$; these have minimal norms $0, \frac{5}{2}, 4, \frac{9}{2}, 4, \frac{5}{2}$. So we must combine the integral ones with the trivial coset of E_7 in E_7^*, and the half-integral ones with the nontrivial coset. In each case the sum of the minimal norms is at least 4 so no new roots appear.

[*Remark*: At this point we can already classify the even unimodular lattices L of rank 17 and discriminant 2: they are complements of E_7 in a Niemeier lattice N, necessarily with $R(N)$ containing an E_7 or E_8 component. There are four: $N = E_8^3$ and $N = D_{16}^+ \oplus E_8$ yield $L = A_1 \oplus E_8^2$ and $A_1 \oplus D_{16}^+$, while the lattices with $R(N) = D_{10} \oplus E_7^2$ and $A_{17} \oplus E_7$ yield $L = (D_{10} \oplus E_7)^+$ and A_{17}^{+3}.]

$\underline{h = 16}$: Here $R(N) = A_{15} \oplus D_9$, which is similar to the case $A_{17} \oplus E_7$ we just saw. The A_{15} cosets must be in $A_{15} + 2A_{15}^*$, and have minimal norms $0, \frac{7}{4}, 3, \frac{15}{4}, 4, \frac{15}{4}, 3, \frac{7}{4}$. The minimal norms for D_9^*/D_9 are $0, \frac{9}{4}, 1, \frac{9}{4}$ (in an order that identifies the discriminant group with $\mathbf{Z}/4\mathbf{Z}$). We need an 8-element subgroup of R^*/R, so we must use the graph of a homomorphism from the 8-element subgroup of A_{15}^*/A_{15} to D_9^*/D_9 (else some element of A_{15}^*/A_{15} would be paired with both the $\tilde{q} = 0$ and the $\tilde{q} = 1$ cosets of D_9 in D_9^*). The integrality condition forces the homomophrism to be surjective; there are two possibilities, and they both work and are equivalent under negating one of the components of R.

$\underline{h = 14}$: Here $R(N) = D_8^3$. In each factor the discriminant group is $(\mathbf{Z}/2\mathbf{Z})^2$, with minimal coset norms $0, 1, 2, 2$. We need an 8-element subgroup of $(D_8^*)^3/D_8^3$ each of whose nonzero elements is of

type $(2, 2, 2)$, $(1, 1, 2)$, or $(0, 2, 2)$ up to permutation (the sum must be even and at least 4). We deduce that this subgroup N/D_8^3 maps surjectively to each component D_8^*/D_8 (else the kernel would be too large to avoid one of the forbidden cosets), and then that it contains a unique triple of each of the three shapes $(2, 1, 1)$, $(1, 2, 1)$, $(1, 1, 2)$. There are 2^3 choices, and each generates a subgroup of the desired type, with the remaining nonzero triples having types $(0, 2, 2)$, $(2, 0, 2)$, $(2, 2, 0)$, and $(2, 2, 2)$. These 2^3 choices are all equivalent under $(\mathrm{Aut}(D_8))^3$ (no need to permute the three D_8 factors), so in this case too there is a unique N up to isomorphism.

$\underline{h = 13}$: Here $R(N) = A_{12}^2$. Each factor A_{12} has discriminant group $\mathbf{Z}/13\mathbf{Z}$, and minimal coset norms $\tilde{q}(a) = a(13-a)/13$ for each a with $0 \leq a \leq 13$. So we need a line in $(\mathbf{Z}/13\mathbf{Z})^2$ consisting of solutions to $a^2 + b^2 = 0$. There are two choices, $b = 5a$ and $b = -5a$, which give rise to isomorphic lattices (for example, by negating one of the two A_{12} factors). The resulting N has the correct $R(N)$ because $\tilde{q}(a) + \tilde{q}(5a) \neq 2$ for all nonzero a (the smallest positive values of $\tilde{q}$ are $12/13$ and $22/13$); in fact $\tilde{q}(a) + \tilde{q}(5a) = 6$ for $a = \pm4, \pm6$ and $\tilde{q}(a) + \tilde{q}(5a) = 4$ for other nonzero $a \in \mathbf{Z}/13\mathbf{Z}$.

$\underline{h = 12}$: There are two cases, including the one possibility $R(N) = A_{11}D_7E_6$ with three non-isomorphic components. That one is actually easy: the discriminant groups of A_{11}, D_7, E_6 are cyclic of order $12, 4, 3$, and we soon see that we must use a cyclic subgroup of $R(N)^*/R(N)$ generated by $\alpha + \delta + \epsilon$ for some generators α, δ, ϵ of A_{11}^*/A_{11}, D_7^*/D_7, E_6^*/E_6 respectively. All choices are equivalent under sign changes of the components of $R(N)$. The $\tilde{q}$ values for nonzero multiples are as follows:

n	1	2	3	4	5	6	7	8	9	10	11
$\tilde{q}(n\alpha)$	$11/12$	$5/3$	$9/4$	$8/3$	$35/12$	3	$35/12$	$8/3$	$9/4$	$5/3$	$11/12$
$\tilde{q}(n\delta)$	$7/4$	1	$7/4$	0	$7/4$	1	$7/4$	0	$7/4$	1	$7/4$
$\tilde{q}(n\epsilon)$	$4/3$	$4/3$	0	$4/3$	$4/3$	0	$4/3$	$4/3$	0	$4/3$	$4/3$
$\sum$	4	4	4	4	6	4	6	4	4	4	4

so indeed the gluing yields an even unimodular lattice and introduces no new roots. For $R(N) = E_6^4$, we need a 9-element subgroup of $R^*/R = (\mathbf{Z}/3\mathbf{Z})^4$ each of whose nonzero elements has exactly 3 nonzero coordinates. That's exactly a maximal isotropic subspace with respect to the quadratic form $\sum_{j=1}^4 x_j^2$; there are eight choices,[4] all equivalent under sign changes (with no need to permute the four E_6 components) to the span of $(0, 1, 1, 1)$ and $(1, 0, 1, -1)$.

[*Remarks*: At this point we can also classify the even unimodular lattices L of rank 18 and discriminant 3: they are complements of E_6 in a Niemeier lattice N, necessarily with $R(N)$ containing an E_6, E_7, or E_8 component. There are six: $N = E_8^3$ and $N = D_{16}^+ \oplus E_8$ yield $L = A_2 \oplus E_8^2$ and $A_2 \oplus D_{16}^+$; the lattices with $R(N) = A_{11}D_7E_6$ and E_6^4 yield $L = (A_{11}D_7)^{+4}$ and $(E_6^3)^{+3}$; and the lattices with $R(N) = D_{10} \oplus E_7^2$ and $A_{17} \oplus E_7$ yield L with $R^+(L) = (D_{10} \oplus E_7)^+$ and A_{17}^{+3}. In each of these E_7 cases, once we know $R^+(L)$, we can recover L in each case as the unique even lattice containing $R^+(L) \oplus \mathbf{Z}\langle 6 \rangle$ with index 2. Here we use $R^+(L)$ for the saturated root lattice $(R(L) \otimes \mathbf{Q}) \cap L$.

Likewise we can extend our classification of *odd* unimodular lattices L to rank $n \leq 17$, via their even sublattices L_0 of discriminant 4, which are orthogonal complements of D_{24-n} in Niemeier lattices N.

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[4] In general, if k is a finite field of order q, and Q is a nondegenerate quadratic form $k^4 \to k$ that has a 2-dimensional isotropic subspace, then the number of such subspaces is $2(q+1)$; these come in two families, corresponding to the two rulings of the smooth quadric $Q = 0$ in the projective space $\mathbf{P}^3(k)$. The case $k = \mathbf{Z}/7\mathbf{Z}$ of this arises for $R(N) = A_6^4$, where there are 16 isotropic subspaces but they still form single orbit under the action of $\{\pm1\}^4 \rtimes S_4$.

Thus if $n \leq 7$ then the only N that contains D_{24-n} is D_{24}^+, giving $L_0 \cong D_n$ (including the cases $n = 1, 2, 3$ in which this lattice has another name) and $L \cong \mathbf{Z}^n$. For $n \geq 9$, we can also put D_{24-n} in the D_{16} component of[5] $D_{16}^+ E_8$, giving $L_0 = D_{n-8} E_8$ and $L = \mathbf{Z}^{n-8} E_8$. (Note that $n = 8$ is not allowed here because D_{16} is not a *saturated* sublattice of $D_{16}^+ E_8$.) For $n \geq 12$, we can also use the D_{12}^2 Niemeier lattice, getting $L = \mathbf{Z}^{n-12} D_{12}^+$; the further lattices $L = \mathbf{Z}^{n-14}(E_7^2)^+$ and $\mathbf{Z}^{n-15} A_{15}^{+4}$ that we found for $n \geq 14$ and $n \geq 15$ correspond to the Niemeier lattices with $R(N) = D_{10} E_7^2$ and $A_{15} D_9$ respectively. At $n = 16$ a new lattice appears, using N with $R(N) = D_8^3$; here $L_0 = (D_8^3)^+$ and L is the unique odd lattice between L_0 and L_0^* (the others being the even unimodulars E_8^2 and D_{16}^+). There are three more possibilities for $n = 17$: if $R(N) = A_{11} D_7 E_6$ then $L_0 = (A_{11} E_6)^{+3}$, and we can also put D_7 in an E_8 factor, so can take $N = E_8^3$, or again $N = D_{16}^+ E_8$ but with an inequivalent embedding of D_7. For each n, the lattice L is new if it's not $\mathbf{Z} \oplus L_1$ for some *odd* unimodular L_1.

Once we complete the Niemeier classification we'll be able to do this for all $n \leq 23$, and also to classify the even lattices with $(\mathrm{rank}, \mathrm{disc}) = (22, 3)$ and $(23, 2)$ and to answer many other such questions.]

The case of E_6^4 is the first example where R has enough isomorphic components that the problem of finding a suitable subgroup G of R^*/R has a coding-theoretic flavor; in this case our isotropic subspace of $R^*/R = (\mathbf{Z}/3\mathbf{Z})^4$ is called the (ternary) "tetracode", and can also be used to find E_8 between A_2^4 and $(A_2^4)^*$. As we go further down the list of possible R, we must take more care not to use a nonzero element of R^*/R that lifts to a new root. This condition roughly corresponds to the requirement that each nonzero element in G have large enough "(Hamming) weight" (in general, the Hamming weight of a vector is its number of nonzero coordinates). We hedge with "roughly" because for most simple root lattice R_1 the nonzero elements of R_1^*/R_1 do not all have the same $\tilde{q}$. Fortunately for $h = 3$ and $h = 2$ the nonzero $\tilde{q}$ values *are* all the same, so the problem of recovering N from R translates directly to a coding-theory problem.

For $R = A_2^{12}$, identify R^*/R with k^{12} where $k = \mathbf{Z}/3\mathbf{Z}$. We seek a subspace $G \subset k^{12}$ of dimension 6 that is maximal isotropic with respect to the quadratic form $\sum_{j=1}^{12} x_j^2$, and moreover contains no vectors of weight exactly 3. It is known that such G exists and is unique up to isomorphism, and furthermore that its automorphism group is the nontrivial double cover of the sporadic Mathieu group M_{12}. (The automorphisms of G are the signed coordinate permutations that take G to itself, and thus lift to automorphisms of N; we are asserting that $\mathrm{Aut}(G)$ has center $\{\pm 1\}$ with $\mathrm{Aut}(G)/\{\pm 1\} = M_{12}$ but $\mathrm{Aut}(G) \not\cong \{\pm 1\} \times M_{12}$. This G is the *extended ternary Golay code* $\mathcal{G}_{12}$. One way to construct it is via the elements of *maximal* weight, that is, with *no* zero coordinates: there are 12 pairs $\{x, -x\}$, and any two are orthogonal even when lifted to (± 1)-vectors over $\mathbf{R}$ (else their sum or difference would have weight 3), so these vectors form a *Hadamard matrix* of order 12, which is known to be unique up to automorphism. One choice of signs that reveals some of the $\{\pm 1\}.M_{12}$ symmetry of H_{12} is

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[5]We drop the "⊕" for the rest of this paragraph (except the final sentence).

$$H_{12} = \begin{pmatrix} + & + & + & + & + & + & + & + & + & + & + & + \\ + & - & - & + & + & - & - & - & + & + & + & - & + \\ + & + & - & - & + & + & - & - & - & + & + & + & - \\ + & - & + & - & - & + & + & - & - & - & + & + & + \\ + & + & - & + & - & - & + & - & - & - & + & + \\ + & + & + & - & + & - & - & + & - & - & - & + \\ + & + & + & + & - & + & - & - & + & - & - & - \\ + & - & + & + & + & - & + & - & - & + & - & - \\ + & - & - & + & + & + & - & + & - & - & + & - \\ + & - & - & - & + & + & + & - & + & - & - & + \\ + & + & - & - & - & + & + & + & - & + & - & - \\ + & - & + & - & - & - & + & + & + & - & + & - \end{pmatrix}, \tag{3}$$

where the first row and column are each $+1$, and for $i, j > 1$ the (i, j) entry is $+1$ if and only if $i - j$ is a nonzero square mod 11. The rows of H_{12} mod 3 generate $\mathcal{G}_{12}$; for example, the first six rows form a basis. The preimage of $\mathcal{G}_{12}$ in $(A_2^*)^{12}$ is thus the unique Niemeier lattice N with $R(N) = A_2^{12}$.

For $R = A_1^{24}$, identify R^*/R with k^{24} where $k = \mathbf{Z}/2\mathbf{Z}$. Now we seek a subspace $G \subset k^{24}$ of dimension 12 that is maximal isotropic with respect to the pairing $\sum_{j=1}^{24} x_j^2$, such that each vector of G has a *doubly even* weight, which is not exactly 4. The doubly-even condition corresponds to N being not just self-dual but also even. Since G is isotropic for the pairing, $(x, x) = \sum_{j=1}^{24} x_j^2 = 0$ for all $x \in G$; but in characteristic 2 this is equivalent to $\sum_{j=1}^{24} x_j = 0$, so $(x, 1^{24}) = 0$ and x has even weight. But if x, y are binary vectors of even weight then $\mathrm{wt}(x + y) - \mathrm{wt}(x) - \mathrm{wt}(y) \equiv 2(x, y) \bmod 4$ (this makes sense because $(x, y) \in \mathbf{Z}/2\mathbf{Z}$). So, on the even-weight vectors, the map $x \mapsto \frac{1}{2}(\mathrm{wt}(x)) \bmod 2$ is a quadratic form that refines the pairing $(\cdot, \cdot)$, and we are asking that G be maximal isotropic for this quadratic form. As was the case in $(\mathbf{Z}/3\mathbf{Z})^{12}$, it is known that there exists G satisfying our conditions (that it be self-dual, doubly even, and without vectors of norm 4), and that this G is unique up to isomorphism. Again a sporadic Mathieu group arises, here the largest one: $\mathrm{Aut}(G) = M_{24}$. This G is the *extended binary Golay code* $\mathcal{G}_{24}$. One choice of basis for $\mathcal{G}_{24}$ is

$$G_{24} = \begin{pmatrix} 1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1 & 1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1 \\ 0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & 1\,1\,0\,1\,0\,0\,0\,1\,1\,1\,0\,1 \\ 0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0 & 1\,1\,1\,0\,1\,0\,0\,0\,1\,1\,1\,0 \\ 0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0 & 1\,0\,1\,1\,0\,1\,0\,0\,0\,1\,1\,1 \\ 0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0 & 1\,1\,0\,1\,1\,0\,1\,0\,0\,0\,1\,1 \\ 0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0 & 1\,1\,1\,0\,1\,1\,0\,1\,0\,0\,0\,1 \\ 0\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0 & 1\,1\,1\,1\,0\,1\,1\,0\,1\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0 & 1\,0\,1\,1\,1\,0\,1\,1\,0\,1\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0 & 1\,0\,0\,1\,1\,1\,0\,1\,1\,0\,1\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0 & 1\,0\,0\,0\,1\,1\,1\,0\,1\,1\,0\,1 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0 & 1\,1\,0\,0\,0\,1\,1\,1\,0\,1\,1\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1 & 1\,0\,1\,0\,0\,0\,1\,1\,1\,0\,1\,1 \end{pmatrix}, \tag{4}$$

with the second 12×12 block having 1's on the diagonal and on the locations of the $+1$ entries of H_{12}. The preimage of $\mathcal{G}_{24}$ in $(A_1^*)^{24}$ is thus the unique Niemeier lattice N with $R(N) = A_1^{24}$.

Naturally SPLAG has much information about $\mathcal{G}_{12}$, $\mathcal{G}_{24}$, and their automorphism groups; for the codes, start with pages 84ff., and for the Mathieu groups, start with the first two of Conway's "Three Lectures on Exceptional Groups" = SPLAG chapter 10, especially sections 1.6 for M_{12} and 2.1ff. for M_{24}.

Finally, consider the case $h = 0$ where the lattice has no roots. Leech constructed such a lattice as a "2-neighbor" of the Niemeier lattice N with $R(N) = A_1^{24}$. Write N as $\{x \in \mathbf{Z}^{24} : x \bmod 2 \in \mathcal{G}_{24}\}$ with inner product $\langle x, y \rangle = \frac{1}{2} \sum_{j=1}^{24} x_j y_j$. Note that $\sum_{j=1}^{24} x_j \equiv 0 \bmod 2$ for all such x. Let N_0 be the index-2 sublattice with $\sum_{j=1}^{24} x_j \equiv 0 \bmod 4$. Then N_0 has minimal norm 4 and discriminant group $N_0^*/N_0 \cong (\mathbf{Z}/2\mathbf{Z})^2$, generated by the cosets of $2e_0$ and $\frac{1}{2} \sum_{j=1}^{24} e_j$. These cosets have minimal norms 2 and 3 respectively, but their sum has minimal norm 4, attained for instance by the vector $2e_0 - \frac{1}{2} \sum_{j=1}^{24} e_j$ of norm $(3^2 + 23 \cdot 1^2)/8 = 4$. Hence this coset together with N_0 constitutes an even self-dual lattice Λ with minimal norm 4.

As part of his classification, Niemeier showed that Λ is the unique such lattice. Conway gave a simpler proof using theta functions, starting from the following calculation. The theta function of any such lattice L must be a linear combination of E_4^3 and Δ whose q^0 and q^1 coefficients are 1 and 0 respectively; hence

$$\theta_L = \mathsf{E}_4^3 - 720\Delta = 1 + 196560q^2 + 16773120q^3 + 398034000q^4 + \cdots . \tag{5}$$

The q^2, q^3, q^4 coefficients are the counts of lattice vectors of norm $4, 6, 8$ respectively, and it turns out that

$$1 + \frac{1}{2}196560 + \frac{1}{2}16773120 + \frac{1}{48}398034000 = 16777216 = 2^{24} = \#(L/2L). \tag{6}$$

As with E_8, it follows[6] that each nonzero coset of $2L$ in L is represented by a pair of vectors of norm 4 or 6, or an orthogonal frame of 24 pairs of vectors of norm 8. In particular, any choice of such a frame gives an embedding of $D_{24}\langle 2 \rangle$ into L; so again we have a sublattice of finite index, reducing the question to a finite combinatorial problem. This problem turns out to be much the same as reconstructing N from $R(N)$ when $R(N) = A_1^{24}$, so Conway recovers Leech's construction of Λ, and also shows that all $398034000/48 = 8292375 = 3^6 5^3 7 \cdot 13$ frames are equivalent, which he then uses to compute $\#(\mathrm{Aut}(\Lambda))$.

At the end of Venkov's paper, he remarks that "Conway's proof is very pretty and can hardly be improved. However, if we assume (as is natural in our approach) that the classification of [Niemeier] lattices with nontrivial root systems is already known, then the characterization of Leech's lattice can be obtained as follows", and then shows how to replace the first step (that is, the step using the calculation in (5,6)) by an argument that in effect reverses Leech's construction. Fix one of the 398034000 lattice vectors u with $\langle u, u \rangle = 8$. Then $u \notin 2L$ because $N_2(L) = 0$. Let N_0 be the index-2 sublattice $\{v \in L : \langle u, v \rangle \equiv 0 \bmod 2\}$. Then $u/2 \in N_0^*$, so $N := N_0 \cup (N_0 + u/2)$ is an even unimodular lattice. It has at least one root, namely $u/2$; and any two roots v, v' must be either proportional or orthogonal, else one of $v \pm v'$ is also a root, which is not possible because $v, v' \in N_0 + u/2$ so $v \pm v' \in N_0$ — but N_0 has no roots. Therefore $R(N) = A_1^{24}$, and we already showed (modulo the characterization of $\mathcal{G}_{24}$) that this determines N uniquely up to isomorphism. Then N_0 is an index-2 sublattice of N containing none of the roots, so N_0 is the kernel of a homomorphism $f : N \to \mathbf{Z}/2\mathbf{Z}$ that maps every root to 1. This determines f on A_1^{24}, so there are $[N : A_1^{24}] = 2^{12}$ such homomorphisms. We claim that each of them is $a \mapsto \sum_{j=1}^{24} \epsilon_j a_j \bmod 2$ for some choice of $\epsilon_j \in \{\pm 1\}$. Proof: there are 2^{24} choices for the ϵ_j, and any two yield the same f if and only if they differ by some element of $(-1)^{\mathcal{G}_{24}}$. So we have 2^{12} distinct f, and our list is complete. Moreover all such f are equivalent under coordinate reflections, so they yield isomorphic N_0 and L and we're done.

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[6]Naturally the argument we gave for E_8, starting from $1 + \frac{1}{2}240 + \frac{1}{16}2160 = 2^8$, was suggested by Conway's uniqueness proof for Λ.

In general, two lattices L, L' in the same genus that can be realized as different index p superlattices of the same L_0 are said to be "p-neighbors"; thus Leech constructed Λ_{24} as a 2-neighbor of the lattice $\{x \in \mathbf{Z}^{24}\langle 1/2 \rangle : x \bmod 2 \in \mathcal{G}_{24}\}$, which we now call the A_1^{24} Niemeier lattice. It is known that in many cases, including the even self-dual lattices of given rank and $p = 2$, all lattices in a genus can be connected by a sequence of 2-neighbor steps. Niemeier obtained his classification by starting from one even unimodular lattice and repeatedly using 2-neighbor steps until each lattice's 2-neighbors were all accounted for. The 2-neighbors of an even lattice L of odd discriminant are classified by the nontrivial cosets in $L/2L$ consisting of vectors of norm $0 \bmod 4$; thus if L has rank n then its number of 2-neighbors is about 2^{n-1}. Fortunately it is enough to try one representative of each $\mathrm{Aut}(L)$-orbit, and for a Niemeier lattice there are at most a few dozen orbits. Even so, computing (by hand!) neighbor associated to each orbit, and checking whether it is isomorphic to an already known lattice, is a substantial computation.

Once we have the list, its completeness can also be confirmed using the Minokwski-Siegel mass formula, which predicts the sum of $1/\#(\mathrm{Aut}(L))$ (total "mass") over all even unimodular lattices L of a given rank: once we have found enough terms $1/\#(\mathrm{Aut}(L))$ to reach the expected mass, there can be no further L. Conway published this computation for the Niemeier lattices in 1982; see SPLAG pages 412–413 (which also acknowledges a paper of Erokhin that appeared in 1979 in Russian and in 1981 in English). Naturally one must exercise care in getting each $\#(\mathrm{Aut}(L))$ exactly right in order to be rewarded with striking confirmation of a sum $\sum_L 1/\#(\mathrm{Aut}(L))$ that agrees with the prediction:

$$\frac{1027637932586061520960267}{129477933340026851560636148613120000000} = \frac{|B_{12}|}{24} \prod_{j=1}^{11} \frac{|B_{2j}|}{4j}, \tag{7}$$

where B_n is the n-th Bernoulli number, so for example $B_2, B_4, B_6, \cdots, B_{16}$ are

$$\frac{1}{6}, \quad -\frac{1}{30}, \quad \frac{1}{42}, \quad -\frac{1}{30}, \quad \frac{5}{66}, \quad -\frac{691}{2730}, \quad \frac{7}{6}, \quad -\frac{3617}{510}. \tag{8}$$

See SPLAG p.412–413 for a sketch of yet another proof (Vinberg, 1972) via the connection between Λ_{24} and the *indefinite* even self-dual lattice $\mathrm{II}_{25,1}$ — though this, too, comes with a warning that "in no sense is this a short-cut to Niemeier's result, for the proofs [of the needed facts about the 'deep holes' in Λ_{24}] require extensive computations" as well.