

Math 272y: Rational Lattices and their Theta Functions

October 23: Some applications of the 7-design property for E_8

Each shell of the E_8 lattice is a spherical 7-design; this follows either from modularity of weighted theta functions (there is no nonzero cusp form of weight less than 12) or from the action of $\text{Aut}(E_8)$ (every invariant homogeneous polynomial of degree < 8 is a polynomial in the norm). We apply this to the first shell, which consists of the 240 roots of E_8 , obtain refined combinatorial and geometric information about the E_8 lattice and its configuration of roots. This can also be used to give several further arguments for the uniqueness of the E_8 lattice and the size of $\text{Aut}(E_8)$ — assuming we do use only modular forms, not the invariants of $\text{Aut}(E_8)$, to prove the 7-design property in the first place.

Our key tool is a formula for the average of x_1^d over any sphere about the origin in \mathbf{R}^n . For now we need $n = 8$, but we shall later use other $n \geq 2$ as well.

Proposition. *Let n, d be integers with $d \geq 0$ and $n \geq 2$. For any unit vector $e \in \mathbf{R}^n$, and any real $r > 0$, the average of $\langle v, e \rangle^d$ over the sphere $\langle v, v \rangle = r^2$ is*

$$r^d \prod_{j=0}^{\frac{d}{2}-1} \frac{2j+1}{n+2j} = \frac{1}{n} \frac{3}{n+2} \frac{5}{n+4} \cdots \frac{d-1}{n+d-2} r^d \quad (1)$$

if d is even, and 0 if d is odd.

The proof is a standard calculation, which we defer to the last page in order to get to our applications more quickly. For now we note that the average must be proportional to r^d , and must vanish for odd d by antisymmetry, so we need only confirm the factor $\prod_{j=0}^{\frac{d}{2}-1} (2j+1)/(2n+j)$ in (1). For $d = 0$ and $d = 2$ we can do this without integration. Thus for $d = 0$ we are averaging the constant function 1, and indeed (1) gives 1 in that case; for $d = 2$, let e_1, \dots, e_n be an orthonormal basis, and note that each $\langle v, e_i \rangle^2$ has the same average while $\sum_{i=1}^n \langle v, e_i \rangle^2 = \langle v, v \rangle = r^2$ on our sphere, whence each $\langle v, e_i \rangle^2$ must average to r^2/n , again in agreement with (1). For $d \geq 4$, see the Proposition's proof at the end of this chapter of the notes.

Now fix some nonzero $v_0 \in E_8$, and let $e = v_0/|v_0|$. Since E_8 is integral, $\langle v_0, e \rangle = |v_0|^{-1} \langle v_0, v \rangle \in |v_0|^{-1} \mathbf{Z}$ for each E_8 root v . Let N_k ($k \in \mathbf{Z}$) be the number of roots with $\langle v_0, v \rangle = k$. Then $N_{-k} = N_k$ for all k , and $N_k = 0$ if $k^2 > 2\langle v_0, v_0 \rangle$ by Cauchy-Schwarz. If $\langle v_0, v_0 \rangle < 8$ then this together with the 7-design property gives us enough conditions to determine each N_k . For example, if v_0 is itself a root (so $|v_0|^2 = 2$), then each $N_{\pm 2} = 1$ because the only v is $\pm v_0$, and we have

$$N_0 + 2N_1 + 2N_2 = \sum_{k=-2}^2 N_k = 240 \quad (2)$$

and

$$2N_1 + 8N_2 = \sum_{k=-2}^2 k^2 N_k = 2 \sum_{\substack{v \in E_8 \\ \langle v, v \rangle = 2}} \langle v, e \rangle^2 = 2 \cdot 2 \cdot \frac{1}{8} \cdot 240 = 120 \quad (3)$$

by the 2-design condition and the case $d = r^2 = 2$ of Proposition 1. This gives us three independent linear equations in the three unknowns N_0, N_1, N_2 ; the solution is

$$(N_0, N_1, N_2) = (126, 56, 1). \quad (4)$$

As a check on this computation we can verify the fourth and sixth moments: for each even $d \geq 2$ we have $\sum_{k=-2}^2 k^d N_k = 2(56 + 4^d)$, which is 144 for $d = 4$ and 240 for $d = 6$, again in agreement with (1) (multiply the second moment 120 by $4 \cdot (3/10)$ to get 144, and multiply this fourth moment by $4 \cdot (5/12)$ to get 240).

In particular, the orthogonal complement of v_0 in E_8 is an even lattice with 126 roots, so it must be E_7 because no other root lattice of rank ≥ 7 has Coxeter number $h \geq 18$ (and because $\text{disc } E_7 = 2$ so an integral lattice cannot properly contain E_7 with finite index). Thus E_8 contains $E_7 \oplus \mathbf{Z}v_0$ with index 2. We already know that there is a unique choice of gluing of E_7 to A_1 to reconstitute E_8 . This proves that $\text{Aut}(E_8)$ acts transitively on the 240 vectors v_0 and that $|\text{Aut}(E_7)| = (2/240) |\text{Aut}(E_8)|$.

Now let v_0 be one of the 2160 vectors of norm 4, so $e = v_0/2$. Then $\lfloor (2|v_0|^2)^{1/2} \rfloor = \lfloor \sqrt{8} \rfloor = 2$, so $|\langle v, v_0 \rangle|$ is still bounded by 2. We do not get $N_{\pm 2}$ for free this time, but still have more than enough linear equations to determine the N_k : again $N_0 + 2N_1 + 2N_2 = 240$, and also

$$2N_1 + 8N_2 = (2 \cdot 4) \cdot \frac{1}{8} \cdot 240 = 240, \quad 2N_1 + 32N_2 = (2 \cdot 4)^2 \cdot \frac{1}{8} \cdot \frac{3}{10} \cdot 240 = 576. \quad (5)$$

Here the solution is

$$(N_0, N_1, N_2) = (84, 64, 14) \quad (6)$$

[and again the sixth moment checks out: $2(64 + 4^3 14) = 1920 = 8 \cdot (5/12) 576$]. Here too we identify the orthogonal complement by its number of roots: there are $N_0 = 84$, so E_7 has too many roots, while the next-smaller Coxeter number for rank ≥ 7 is $12 = 84/7$ which is attained only by E_6 and D_7 ; so $v_0^\perp \cap E_8 \cong D_7$, and again we can reconstruct E_8 as the unique gluing, this time up to reflection about the D_7 hyperplane. As a further check, $N_{\pm 1} = 64$ is the number of dual vectors of minimal norm $7/4$ in each of the non-integral cosets of D_7 in D_7^* .

In fact, in this case we do not need the classification of root systems: we can show directly that the 14 vectors counted by N_2 , together with v_0 , generate a copy of D_8 in E_8 ; and we have already seen that this determines E_8 uniquely and proves the transitivity of the action of $\text{Aut}(E_8)$ on the norm-4 lattice vectors. We begin by observing that the vectors v with $\langle v, v \rangle = \langle v, v_0 \rangle = 2$ pair up under $v \longleftrightarrow v_0 - v$. This can be seen by direct computation, or by writing $v = v' + e$ (recall that $e = v_0/2$) with $\langle v', e \rangle = 0$ and $\langle v', v' \rangle = 1$. Now if $v_1 = v'_1 + e$ and $v_2 = v'_2 + e$ are two such vectors then $\langle v'_1, v'_2 \rangle = \langle v_1, v_2 \rangle - \langle e, e \rangle = \langle v_1, v_2 \rangle - 1 \in \mathbf{Z}$. By Cauchy-Schwarz, then, either $v'_2 = \pm v'_1$ or $\langle v'_1, v'_2 \rangle = 0$. Hence the 14 vectors v' constitute an orthonormal frame for v_0^\perp , and those vectors together with $\pm e$ form an orthonormal frame for \mathbf{R}^8 . In the coordinates of this frame it is easy to see that v_0 and the vectors $v_i - v_j$ (for any roots v_i, v_j with $\langle v_i, v_0 \rangle = \langle v_j, v_0 \rangle = 2$) generate a copy of D_8 in E_8 , and the rest we have done already.

Using also the sixth moment, we can still give a similar treatment of the next two cases, with $\langle v_0, v_0 \rangle = 6$ and $\langle v_0, v_0 \rangle = 8$. In the first case, $\lfloor (2|v_0|^2)^{1/2} \rfloor = \lfloor \sqrt{12} \rfloor = 3$, and the equations are

$N_0 + 2N_1 + 2N_2 + N_3 = 240$ and $2 \sum_{k=1}^3 k^{2d} N_k = 360, 1296, 6480$ for $d = 1, 2, 3$, with solution

$$(N_0, N_1, N_2, N_3) = (74, 54, 27, 2). \quad (7)$$

This time there are only two vectors attaining the maximal $\langle v, v_0 \rangle$, and again they must be related by $v \longleftrightarrow v_0 - v$; so we have shown that *every E_8 vector of norm 6 is uniquely a sum of two roots*, necessarily with inner product 1. (Check: the number of such pairs of roots is $\frac{1}{2}240 \cdot 56 = 240 \cdot 28 = (1^3 + 3^3)240$, which is the q^3 coefficient of the normalized Eisenstein series $\bar{E}_4 = \theta_{E_8}$.) Now those two vectors generate a copy of A_2 inside E_8 , and their differences $\pm(2v - v_0)$ are among the 74 roots orthogonal to v_0 . The remaining 72 must be orthogonal not just to v_0 but to the entire A_2 copy (the projections to $A_2 \otimes \mathbf{R}$ land in A_2^* , which has no sufficiently short nonzero vectors orthogonal to v_0); so the orthogonal complement of this A_2 must be isomorphic with E_6 , the only root lattice of rank at most 6 with as many as 72 roots. Again we reconstruct E_8 by gluing A_2 to E_6 , and deduce that all rank-6 vectors in E_8 are equivalent under $\text{Aut}(E_8)$, as are all embeddings $A_2 \hookrightarrow E_8$.

For $\langle v_0, v_0 \rangle = 8$, there are $(1^3 + 2^3 + 4^3)240 = 73 \cdot 240$ possible v_0 , so they cannot constitute a single orbit under $\text{Aut}(E_8)$. But there is an easy invariant: 240 of these v_0 are of the form $2v$ for some root v . We shall see that the remaining $72 \cdot 240$ are equivalent. Here $\lfloor (2|v_0|^2)^{1/2} \rfloor = \lfloor \sqrt{16} \rfloor = 4$, but $\langle v, v_0 \rangle = \pm 4$ can happen only for $v = \pm v_0/2$, so if v is not a double root we again have only four unknowns N_0, N_1, N_2, N_3 and just enough linear equations to solve. We find

$$(N_0, N_1, N_2, N_3) = (56, 56, 28, 8). \quad (8)$$

Since the orthogonal complement of v_0 has 56 roots in this case, it must be A_7 , and again we can reconstruct E_8 by gluing, this time of the discriminant-8 lattices A_7 and $\mathbf{Z}v_0$. But here we don't need the classification of root systems because we can use $N_3 = 8$: the configuration of eight roots with $\langle v, v_0 \rangle = 3$ is uniquely determined — any two have inner product 1 — and they generate a lattice A_8 , from which we recover the full E_8 lattice (necessarily the same as A_8^{+3}) by including the translates by $\pm v_0$. Indeed each projection $v - \frac{3}{8}v_0$ has norm $7/8$, and any two of them have inner product $\langle v - \frac{3}{8}v_0, v' - \frac{3}{8}v_0 \rangle = -1/8$ (it must be $\equiv 7/8 \pmod{1}$ and of absolute value $\leq 7/8$); so $\langle v, v' \rangle = -1/8 + (3/8)^2 \langle v_0, v_0 \rangle = 9/8 - 1/8 = 1$. The 56 differences $v - v'$ are the roots of the orthogonal complement of v_0 . Each of the other N_k , namely $\binom{8}{3}, \binom{8}{2}, \binom{8}{1}$, count vectors of minimal norm $15/8, 12/8, 7/8$ in a coset of A_7 in A_7^* .

Somewhat surprisingly, we can even handle $\langle v_0, v_0 \rangle = 10$ and $\langle v_0, v_0 \rangle = 12$, even though in these cases there are five variables N_0, N_1, N_2, N_3, N_4 and still only four equations. This is thanks to the additional condition that each N_k must be a nonnegative integer.

For $\langle v_0, v_0 \rangle = 10$, the general solution is¹

$$(N_0, N_1, N_2, N_3, N_4) = (70N_4 - 10, 100 - 56N_4, 28N_4 + 5, 20 - 8N_4, N_4), \quad (9)$$

¹The N_4 coefficients $70, -56, 28, -8, 1 = (-1)^k \binom{8}{4-k}$ of N_k (and also for $k = -1, -2, -3, -4$) can be explained by noting that $\sum_{k=-4}^4 (-1)^k \binom{8}{4-k} P(k) = 0$ for all polynomials P with $\deg P < 8$, which in turn holds because $\sum_{k=-4}^4 (-1)^k \binom{8}{4-k} f(k)$ is an 8-th finite difference of f .

and nonnegativity of N_0 and N_1 implies $1/7 \leq N_4 \leq 25/14$; since $N_4 \in \mathbf{Z}$ it follows that $N_4 = 1$ and

$$(N_0, N_1, N_2, N_3, N_4) = (60, 44, 33, 12, 1). \quad (10)$$

If v is the unique root such that $\langle v, v_0 \rangle = 4$ then $v' := v_0 - 2v$ has norm $10 - 16 + 8 = 2$ and $\langle v, v' \rangle = 0$; and conversely every decomposition $v_0 = 2v + v'$ with $\langle v, v \rangle = \langle v', v' \rangle = 2$ (and thus $\langle v, v' \rangle = 0$) makes $\langle v_0, v \rangle = 4$. Therefore, every E_8 vector of norm 10 is uniquely $2v + v'$ for some orthogonal roots v, v' . The $60 = N_0$ roots orthogonal to v_0 are then orthogonal to v, v' as well, and constitute a D_6 root system. As we did for $\langle v_0, v_0 \rangle = 6$, we can check that the number of norm-10 vectors is the same whether one computes it by counting pairs (v, v') (finding $240 \cdot 126$) or from the theta function (finding $(1^3 + 5^3)240$); and this coincidence gives another proof, once we check that it is not possible for two different orthogonal root pairs (v, v') and (w, w') to give the same v_0 .

For $\langle v_0, v_0 \rangle = 12$, the general solution is

$$(N_0, N_1, N_2, N_3, N_4) = (70N_4 - 164, 216 - 56N_4, 28N_4 - 54, 40 - 8N_4, N_4) \quad (11)$$

and this time N_4 must be 3, so

$$(N_0, N_1, N_2, N_3, N_4) = (46, 48, 30, 16, 3). \quad (12)$$

The root number $N_0 = 46$ of the orthogonal complement forces it to be $A_2 \oplus D_5$, which has discriminant 12 and again glues uniquely to $\mathbf{Z}v_0$ (up to $\text{Aut}(A_2 \oplus D_5 \oplus \mathbf{Z}v_0)$) to give E_8 . For example, the $48 = 3 \cdot 16$ vectors counted by N_1 are $(v_0/12) + v_1 + v_2$ where v_1 and v_2 are vectors of minimal norms $2/3$ and $5/4$ in cosets that generate A_2^*/A_2 and D_5^*/D_5 ; the 3 vectors counted by N_4 are $(v_0/3) + v_1$ for the same three vectors v_1 .

Appendix: the average of x_1^d over a sphere. As promised, we conclude with a proof of the Proposition that gives the formula (1) for the average of x_1^d over a radius- r sphere in \mathbf{R}^n when $2|d|$.

Proof of Proposition. For each d , the integral over the sphere $\langle v, v \rangle = r^2$ of $\langle v, e \rangle^d$ is

$$C_n \int_{-r}^r x^d (r^2 - x^2)^{(n-3)/2} dx = C_n r^{d+n-2} \int_{-1}^1 z^d (1 - z^2)^{(n-3)/2} dz \quad (13)$$

for some constant $C_n > 0$ (the surface area of the unit sphere in \mathbf{R}^{n-1}). So we need the quotient of the integral (13) by its value at $d = 0$. The factors C_n cancel out, and the ratio r^{d+n-2}/r^{n-2} gives the expected factor of r^d ; so it remains to evaluate the integral over $|z| < 1$. By symmetry,

$$\int_{-1}^1 z^d (1 - z^2)^{(n-3)/2} dz = (1 + (-1)^d) \int_0^1 z^d (1 - z^2)^{(n-3)/2} dz, \quad (14)$$

and then the change of variable $z^2 = t$ (so $2z dz = dt$) gives

$$\int_{-1}^1 z^d (1 - z^2)^{(n-3)/2} dz = \frac{1 + (-1)^d}{2} \int_0^1 t^{(d-1)/2} (1 - t)^{(n-3)/2} dt. \quad (15)$$

We can now evaluate the integral as a special case of the formula

$$\int_0^1 t^{a-1}(1-t)^{b-1} dt = B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad (a, b > 0) \quad (16)$$

for the Beta function. Here this is $B((d+1)/2, (n-1)/2)$, so the quotient we need is

$$\frac{B((d+1)/2, (n-1)/2)}{B(1/2, (n-1)/2)} = \frac{\Gamma((d+1)/2)/\Gamma((n+d)/2)}{\Gamma(1/2)/\Gamma(n/2)} = \frac{\Gamma((d+1)/2)/\Gamma(1/2)}{\Gamma((n+d)/2)/\Gamma(n/2)}. \quad (17)$$

By the identity $\Gamma(s+1) = s\Gamma(s)$, the numerator $\Gamma((d+1)/2)/\Gamma(1/2)$ is $\prod_{j=0}^{\frac{d}{2}-1} (j + \frac{1}{2})$, and the numerator $\Gamma((n+d)/2)/\Gamma(n/2)$ is $\prod_{j=0}^{\frac{d}{2}-1} (j + \frac{n}{2})$. Removing the common factor $2^{d/2}$ recovers the factor $\prod_{j=0}^{\frac{d}{2}-1} (2j+1)/(2n+j)$ in (1), and we are done. \square