

Math 272y: Rational Lattices and their Theta Functions

16 and 21 October 2019: Harmonic polynomials

We next introduce *weighted theta functions* $\Theta_{L,P}, \theta_{L,P}$ of a lattice $L \subset \mathbf{R}^n$. These generalize the theta functions Θ_L, θ_L : they are generating functions that encode not just the number $N_{2k}(L)$ of lattice vectors of each norm $2k$ but also their distribution on the sphere $\langle x, x \rangle = 2k$.

Weighted theta functions are defined as follows. The *weight* P is a *harmonic polynomial* on \mathbf{R}^n , that is, a homogeneous polynomial whose Laplacian vanishes (we shall give a fuller description of such polynomials soon). Then

$$\Theta_{L,P}(q) := \sum_{v \in L} P(v) q^{\langle v, v \rangle / 2} = \sum_{k \geq 0} N_{2k}(L, P) q^k, \quad (1)$$

$$\theta_{L,P}(z) := \sum_{v \in L} P(v) e^{\pi i \langle v, v \rangle z} = \sum_{k \geq 0} N_k(L, P) e^{\pi i k z} = \Theta_{L,P}(e^{2\pi i z}), \quad (2)$$

where we define

$$N_k(L, P) := \sum_{\substack{v \in L \\ \langle v, v \rangle = k}} P(v). \quad (3)$$

Let $d = \deg(P)$. If $d = 0$ then P is a constant, so $\Theta_{L,P}$ and $\theta_{L,P}$ reduce to scalar multiples of Θ_L and θ_L . If d is odd then $\Theta_{L,P} = \theta_{L,P} = 0$ because the v and $-v$ terms cancel. For even $d > 0$ we get a nontrivial generalization of Θ_L and θ_L ; in this case $N_0(L, P) = 0$ so the sums over k in (1, 2) may be taken over $k > 0$.

The definitions of $\Theta_{L,P}$ and $\theta_{L,P}$ make sense for any polynomial P , harmonic or not. We require that P be harmonic so that we can generalize the functional equation to weighted theta functions. Again we shall prove the functional equation using Poisson summation; here the relevant functions on \mathbf{R}^n are

$$f(x) = P(x) e^{-\pi \langle x, x \rangle t}. \quad (4)$$

Theorem 1. *Suppose that $t > 0$ and P is a harmonic polynomial on \mathbf{R}^n of degree d , and define a function $f : \mathbf{R}^n \rightarrow \mathbf{R}$ by (4). Then the Fourier transform of f is*

$$\hat{f}(y) = i^d t^{-(\frac{n}{2} + d)} P(y) e^{-\pi \langle y, y \rangle / t}. \quad (5)$$

This will yield:

Proposition 1 (functional equation for weighted theta series). *For any lattice L in \mathbf{R}^n , and any harmonic polynomial f of degree d , we have*

$$\Theta_{L^*,P}(e^{-2\pi t}) = i^d \text{disc}(L)^{1/2} t^{-(n/2) - d} \Theta_{L,P}(e^{-2\pi/t}) \quad (6)$$

for all $t > 0$.

Proof (assuming Theorem 1): Apply Poisson summation to L and the function (4), and use the formula (5) for the Fourier transform of this function. Q.E.D.

Note that since $\Theta_{L,P}$ and $\Theta_{L^*,P}$ vanish identically for odd d we can write the factor i^d as $(-1)^{d/2}$. (Theorem 1 is still of use to us for odd d when applied to lattice translates; for example, we can use the case $(n, d) = (1, 1)$ to prove the modularity of

$$q^{1/8}(1 - 3q + 5q^3 - 7q^6 + 9q^{10} - 11q^{15} + 13q^{21} - + \dots)$$

which turns out to be $\eta(z)^3$.)

To prove Theorem 1, and then to use Proposition 1 to study lattices, we next review some key properties of harmonic polynomials.

Let \mathcal{P} be the \mathbf{C} -vector space of polynomials on \mathbf{R}^n , and \mathcal{P}_d ($d = 0, 1, 2, \dots$) its subspace of homogeneous polynomials of degree d , so that $\mathcal{P} = \bigoplus_{d=0}^{\infty} \mathcal{P}_d$. The *Laplacian* is the differential operator¹

$$\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} : \mathcal{C}^{\infty}(\mathbf{R}^n) \rightarrow \mathcal{C}^{\infty}(\mathbf{R}^n), \quad \mathcal{P} \rightarrow \mathcal{P}, \quad \mathcal{P}_d \rightarrow \mathcal{P}_{d-2}. \quad (7)$$

Here x_1, \dots, x_n are any orthonormal coordinates on \mathbf{R}^n , and \mathcal{P}_d is taken to be $\{0\}$ for $d < 0$. The space of harmonic polynomials of degree d is then

$$\mathcal{P}_d^0 := \ker(\Delta : \mathcal{P}_d \rightarrow \mathcal{P}_{d-2}); \quad (8)$$

this is the degree- d homogeneous part of

$$\mathcal{P}^0 := \ker(\Delta : \mathcal{P} \rightarrow \mathcal{P}). \quad (9)$$

Examples. \mathcal{P}_0^0 and \mathcal{P}_1^0 are the spaces of constant and linear functions respectively, of dimensions 1 and n . If $n = 1$ then $\mathcal{P}_d^0 = \{0\}$ for all $d > 1$. If $n = 2$ then \mathcal{P}_d^0 is 2-dimensional for each $d > 0$, generated by the real and imaginary parts of $(x_1 + ix_2)^d$. For any n , A quadratic polynomial $P = \sum_{1 \leq j \leq k \leq n} a_{jk} x_j x_k$ is harmonic if and only if $\sum_{j=1}^n a_{jj} = 0$, because ΔP is the constant polynomial $2 \sum_{j=1}^n a_{jj}$.

We shall see that $\Delta : \mathcal{P}_d \rightarrow \mathcal{P}_{d-2}$ is surjective, whence

$$\dim \mathcal{P}_d^0 = \dim(\mathcal{P}_d) - \dim(\mathcal{P}_{d-2}) = \binom{n+d-1}{d} - \binom{n+d-3}{d-2}. \quad (10)$$

Indeed we shall give a more precise result using two further operators on $\mathcal{C}^{\infty}(\mathbf{R}^n)$ and on its subspace \mathcal{P} . The first is²

$$\mathbf{E} := x \cdot \nabla = \sum_{j=1}^n x_j \frac{\partial}{\partial x_j}. \quad (11)$$

¹The use of Δ for both this operator and the modular form $\eta^{24} = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$ may be unfortunate, but should not cause confusion because the two Δ 's never appear together outside this footnote. The alternative notation L for the Laplacian would be much worse when we regularly use L for a lattice.

²Fortunately this operator will never appear together with an Eisenstein series $\mathbf{E}_{2k} \dots$

Euler proved that if $P \in C^\infty(\mathbf{R}^n)$ is homogeneous of degree d then $EP = d \cdot P$; in particular \mathcal{P}_d is the d -eigenspace of $E|_{\mathcal{P}}$. The second operator is multiplication by the norm:

$$F := \langle x, x \rangle = \sum_{j=1}^n x_j^2, \quad P \mapsto \langle x, x \rangle P. \quad (12)$$

Clearly F injects each \mathcal{P}_d into \mathcal{P}_{d+2} . Thus $\mathcal{P}_d^0 = \ker(F\Delta : \mathcal{P}_d \rightarrow \mathcal{P}_d)$; that is, \mathcal{P}_d^0 is the zero eigenspace of the operator $F\Delta$ on \mathcal{P}_d . We next show that the other eigenspaces are $F^k \mathcal{P}_{d-2k}^0$ for $k = 1, 2, \dots, \lfloor d/2 \rfloor$, and that \mathcal{P}_d is the direct sum of these eigenspaces, from which the surjectivity of $\Delta : \mathcal{P}_d \rightarrow \mathcal{P}_{d-2}$ will follow as a corollary.

We begin with by finding the commutators of Δ, E, F . Recall that the *commutator* of any two operators A, B on some vector space is

$$[A, B] = AB - BA = -[B, A]. \quad (13)$$

For example, $[x_j, x_k] = [\partial/\partial x_j, \partial/\partial x_k] = 0$ for all j, k , while $[\partial/\partial x_j, x_k] = \delta_{jk}$ (Kronecker delta).

Lemma 1 (Commutation relations for Δ, E, F). *We have*

$$[\Delta, F] = 4E + 2n, \quad [E, \Delta] = -2\Delta, \quad [E, F] = +2F. \quad (14)$$

Proof: These are direct computations using the pairwise commutators of the operators x_j and $\partial/\partial x_k$. For example, $[\partial^2/\partial x_j^2, x_k^2] = 0$ unless $j = k$, and then we calculate

$$\begin{aligned} \frac{\partial^2}{\partial x_j^2} \circ x_j^2 &= \frac{\partial}{\partial x_j} \circ \left(x_j \frac{\partial}{\partial x_j} + \left[\frac{\partial}{\partial x_j}, x_j \right] \right) \circ x_j = \left(\frac{\partial}{\partial x_j} \circ x_j \right)^2 + \frac{\partial}{\partial x_j} \circ x_j \\ &= \left(x_j \frac{\partial}{\partial x_j} + 1 \right)^2 + x_j \frac{\partial}{\partial x_j} + 1 = \left(x_j \frac{\partial}{\partial x_j} \right)^2 + 3x_j \frac{\partial}{\partial x_j} + 2 = x_j^2 \frac{\partial^2}{\partial x_j^2} + 4x_j \frac{\partial}{\partial x_j} + 2, \end{aligned}$$

whence

$$\left[\frac{\partial^2}{\partial x_j^2}, x_k^2 \right] = \delta_{jk} \left(4x_j \frac{\partial}{\partial x_j} + 2 \right).$$

Summing over $j, k = 1, \dots, n$ yields $[\Delta, F] = 4E + 2n$ as claimed. We complete the proof of Lemma 1 by verifying the remaining two identities in (14) via a similar but easier calculation. \square

For a further check on the formulas for $[E, \Delta]$ and $[E, F]$, note that they are consistent with the action of Δ, E, F on the \mathcal{P}_d : if $P \in \mathcal{P}_d$ then $\Delta P \in \mathcal{P}_{d-2}$ yields

$$[E, \Delta]P = E\Delta P - \Delta EP = (d-2)\Delta P - \Delta(d \cdot P) = -2\Delta P,$$

consistent with $[E, \Delta] = -2\Delta$, and likewise for the third part $[E, F] = +2F$ of (14).

See the further Remarks at the end of this section for the interpretation of the commutation relations (14) (and Lemmas 3, 4 below) in terms of \mathfrak{sl}_2 and other Lie algebras and groups.

Now suppose $P \in \mathcal{P}_d$ is in the λ -eigenspace of $F\Delta$ for some λ . Then $\langle x, x \rangle P = FP$ is in the $(\lambda + 4d + 2n)$ -eigenspace of $F\Delta$ acting on \mathcal{P}_{d+2} , because

$$F\Delta FP = F(F\Delta + [\Delta, F])P = F(F\Delta + 4E + 2n)P = F(\lambda + 4d + 2n)P.$$

By induction on $k = 0, 1, 2, \dots$ it follows that $F^k P$ is an eigenvector of $F\Delta|_{\mathcal{P}_{d+2k}}$ with eigenvalue

$$\lambda + \sum_{j=0}^{k-1} 4(d + 2j) + 2n = \lambda + k(4(d + k - 1) + 2n).$$

Replacing d by $d - 2k$ and taking $\lambda = 0$, we see that if $P \in \mathcal{P}_{d-2k}^0$ then $F^k P$ is an eigenvector of $F\Delta|_{\mathcal{P}_d}$ with eigenvalue

$$\lambda_d(k) := k(4(d - k - 1) + 2n). \quad (15)$$

We next prove that this accounts for all the eigenspaces of $F\Delta|_{\mathcal{P}_d}$.

Lemma 2. Fix $d \geq 0$. For integers k, k' such that $0 \leq k < k' \leq d/2$ we have $\lambda_d(k) < \lambda_d(k')$.

Proof: By induction it is enough to check this for $k' = k + 1$. We compute

$$\lambda_d(k + 1) - \lambda_d(k) = 2n + 4(d - 2k') \geq 2n > 0,$$

Q.E.D.

Corollary. The sum of the subspaces $F^k \mathcal{P}_{d-2k}^0$ of \mathcal{P}_d over $k = 0, 1, \dots, \lfloor d/2 \rfloor$ is direct.

Proof: By Lemma 2, the $\lambda_d(k)$ are strictly increasing, and thus distinct. Our claim follows because $F^k \mathcal{P}_{d-2k}^0$ is a subspace of the $\lambda_d(k)$ eigenspace of $F\Delta$.

Proposition 2. For $k = 0, 1, \dots, \lfloor d/2 \rfloor$, let $\mathcal{P}_d^k = F^k \mathcal{P}_{d-2k}^0$. Then:

- i) The map $\Delta : \mathcal{P}_d \rightarrow \mathcal{P}_{d-2}$ is surjective.
- ii) $\mathcal{P}_d = \bigoplus_{k=0}^{\lfloor d/2 \rfloor} \mathcal{P}_d^k = \mathcal{P}_d^0 \oplus F\mathcal{P}_{d-2}$, and $\mathcal{P} = \bigoplus_{k=0}^{\infty} F^k \mathcal{P}^0$.
- iii) \mathcal{P}_d^k is the entire $\lambda_d(k)$ eigenspace of $F\Delta|_{\mathcal{P}_d}$, and $F\Delta|_{\mathcal{P}_d}$ has no eigenvalues other than the $\lambda_d(k)$ for $k = 0, 1, \dots, \lfloor d/2 \rfloor$.
- iv) $\dim \mathcal{P}_d^0 = \dim(\mathcal{P}_d) - \dim(\mathcal{P}_{d-2})$ as claimed in (10).

Proof: The sum $\bigoplus_{k=0}^{\lfloor d/2 \rfloor} \mathcal{P}_d^k$ is direct by the previous Corollary. We prove that it equals \mathcal{P}_d by comparing dimensions. Since F is injective we have $\dim \mathcal{P}_d^k = \dim \mathcal{P}_{d-2k}^0$; moreover

$$\dim \mathcal{P}_{d-2k}^0 \geq \dim \mathcal{P}_{d-2k} - \dim \mathcal{P}_{d-2k-2},$$

with equality if and only if $\Delta : \mathcal{P}_{d-2k} \rightarrow \mathcal{P}_{d-2k-2}$ is surjective, because $\mathcal{P}_{d-2k}^0 = \ker(\Delta : \mathcal{P}_{d-2k} \rightarrow \mathcal{P}_{d-2k-2})$. Hence $\dim \bigoplus_{k=0}^{\lfloor d/2 \rfloor} \mathcal{P}_d^k$ is

$$\sum_{k=0}^{\lfloor d/2 \rfloor} \dim \mathcal{P}_d^k = \sum_{k=0}^{\lfloor d/2 \rfloor} \dim \mathcal{P}_{d-2k}^0 \geq \sum_{k=0}^{\lfloor d/2 \rfloor} (\dim \mathcal{P}_{d-2k} - \dim \mathcal{P}_{d-2k-2}), \quad (16)$$

and the last sum telescopes to $\dim \mathcal{P}_d$. Thus equality holds termwise in the last step of (16) and $\dim \bigoplus_{k=0}^{\lfloor d/2 \rfloor} \mathcal{P}_d^k = \dim \mathcal{P}_d$. The first of these proves (i) (using the $k = 0$ term). The second yields

$$\mathcal{P}_d = \bigoplus_{k=0}^{\lfloor d/2 \rfloor} \mathcal{P}_d^k, \quad (17)$$

as claimed in (ii); taking the direct sum over d yields $\mathcal{P} = \bigoplus_{k=0}^{\infty} \mathbb{F}^k \mathcal{P}^0$, also claimed in (ii). To complete the proof of (ii) we compare the decompositions (17) of \mathcal{P}_d and \mathcal{P}_{d-2} and note that $\mathcal{P}_d^k = \mathbb{F} \mathcal{P}_{d-2}^{k-1}$ for each $k > 0$. Claim (iii) follows because the decomposition (17) diagonalizes $\mathbb{F} \Delta|_{\mathcal{P}_d}$. Finally (iv) is the case $k = 0$ of equality in (16), Q.E.D.

Remark: Part (ii) implies $\mathcal{P}_d^0 \cap \mathbb{F} \mathcal{P}_{d-2} = \{0\}$, and thus that \mathcal{P}^0 contains no nonzero multiple of $\langle x, x \rangle$. Proving this was set as problem B-5 on the 2005 Putnam examination, which was the hardest of the 12 problems that year, solved by only five of the top 200 scorers. The solution printed in the report on the contest uses some of the ingredients used here to prove Proposition 2.³

We next characterize the functions $f(x) = P(x) e^{-\pi \langle x, x \rangle t}$ of (4) using the operators Δ, E, F . For $t \in \mathbb{C}$ define an operator

$$G_t : \mathcal{C}^\infty(\mathbb{R}^n) \rightarrow \mathcal{C}^\infty(\mathbb{R}^n), \quad g \mapsto e^{-\pi t \langle x, x \rangle} g \quad (18)$$

that multiplies every \mathcal{C}^∞ function by the Gaussian $e^{-\pi t \langle x, x \rangle}$; these operators constitute a one-parameter group: $G_t G_{t'} = G_{t+t'}$ for all t, t' . We are then interested in $f = G_t P$ for $P \in \mathcal{P}$ in the intersection of the kernel of Δ with an eigenspace of E . If $P \in \mathcal{P}_d$ then

$$d \cdot f = G_t(d \cdot P) = G_t E P = (G_t E G_{-t}) G_t P = (G_t E G_{-t}) f, \quad (19)$$

so f is in the d -eigenspace of $G_t E G_{-t}$; likewise $f \in \ker G_t \Delta G_{-t}$. Since our one-parameter group $\{G_t\}$ has infinitesimal generator $-\pi F$, we expect that conjugation by G_t will take Δ, E to some linear combination of Δ, E, F . Indeed we find:

Lemma 3 (Conjugation of Δ, E, F by G_t). *The operators G_t commute with F , and we have*

$$G_t E G_{-t} = E + 2\pi t F, \quad G_t \Delta G_{-t} = \Delta + \pi t (4E + 2n) + (2\pi t)^2 F. \quad (20)$$

Proof: As with Lemma 2, this comes down to an exercise in differential calculus. Here we start from the fact that G_t commutes with each x_j while $G_t(\partial/\partial x_j)G_{-t} = 2\pi t x_j + (\partial/\partial x_j)$, whence

³See: Klosinski, L.F., Alexanderson, G.L., and Larson, L.C.: The Sixty-Sixth William Lowell Putnam Mathematical Competition, *American Math. Monthly* **113** #8 (October 2006), 733–743. Problem B-5 appears on page 736; the score distribution is on page 741; and the solution is on page 742. Suppose $\langle x, x \rangle | P$ and $\Delta P = 0$. Write $P = \sum_{d \geq 0} P_d$ with each $P_d \in \mathcal{P}_d$. Then $\langle x, x \rangle | P_d$ and $\Delta P_d = 0$ for each d , and we may choose d so that $P_d \neq 0$. Let m be the largest integer such that $P_d = \langle x, x \rangle^m Q$ for some polynomial Q ; by assumption $m > 0$. In our notations, then, $Q \in \mathcal{P}_{d-2m}$ with $\Delta F^m Q = 0$ and $Q \notin \mathbb{F} P$. Using in effect the formula for $[\Delta, F]$ and Euler's description of E , compute that

$$\Delta F^m Q = F^{m-1} [\mathbb{F} \Delta + 2m(n + 2(d - m - 1))] Q$$

for all $Q \in \mathcal{P}_{d-2m}$. Thus $\Delta F^m Q = 0$ implies $\mathbb{F} \Delta Q = -2m(n + 2(d - m - 1))Q$, and the factor $n + 2(d - m - 1)$ is positive (because $d \geq 2m \geq m + 1$), so $Q \in \mathbb{F} P$, contradiction.

the first formula in (20) quickly follows, while $G_t F = F G_t$ is immediate. A somewhat longer computation establishes the second formula. \square

Corollary. *The operators Δ, E, F act on $G_t \mathcal{P}$, and the subspace $G_t \mathcal{P}_d^0$ is the intersection of $\ker(\Delta + \pi t(4E + 2n) + (2\pi t)^2 F)$ with the d -eigenspace of $E + 2\pi t F$ in $G_t \mathcal{P}$.*

We next relate the Fourier transform of a Schwartz function f with the Fourier transforms of its images under Δ, E, F , and use this to prove Theorem 1. Recall that we define the Fourier transform $\hat{f} : \mathbf{R}^n \rightarrow \mathbf{C}$ of a Schwartz function $f : \mathbf{R}^n \rightarrow \mathbf{C}$ by

$$\hat{f}(y) = \int_{x \in \mathbf{R}^n} f(x) e^{2\pi i \langle x, y \rangle} d\mu(x). \quad (21)$$

Lemma 4 (Conjugation of Δ, E, F by the Fourier transform). *Let $f : \mathbf{R}^n \rightarrow \mathbf{C}$ be any Schwartz function. Then:*

- i) *For each $j = 1, \dots, n$, the Fourier transform of $x_j f$ is $(2\pi i)^{-1} \partial \hat{f} / \partial y_j$, and the Fourier transform of $\partial f / \partial x_j$ is $-2\pi i y_j \hat{f}$.*
- ii) *The Fourier transforms of Δf , $(2E + n)f$, and Ff are respectively $-(2\pi)^2 F \hat{f}$, $-(2E + n)\hat{f}$, and $-(2\pi)^{-2} \Delta \hat{f}$.*

Proof: Again this is a calculus exercise, here with definite integrals. The formula for the Fourier transform of $\partial f / \partial x_j$ is obtained by integrating by parts with respect to x_j . The Fourier transform of $x_j f$ can be obtained from this using Fourier inversion, or directly by differentiation with respect to y_j of the integral (21) that defines $\hat{f}(y)$. We then obtain (ii) by iterating the formulas in (i) to find the Fourier transform of $\partial^2 f / \partial x_j^2$, $x_j \partial f / \partial x_j$, or $x_j^2 f$, and summing over j . The case of Ef can be explained by writing $2E + n$ as $\sum_{j=1}^n (x_j (\partial / \partial x_j) + (\partial / \partial x_j) \circ x_j)$, Q.E.D.

We first use this to show that if $f \in G_t \mathcal{P}$ then $\hat{f} \in G_{1/t} \mathcal{P}$, that is, that \hat{f} is *some* polynomial multiplied by $e^{-\pi \langle y, y \rangle / t}$; more precisely:

Proposition 3. *Let $t \in \mathbf{C}$ with $\operatorname{Re}(t) > 0$. If $f = G_t P$ for some $P \in \mathcal{P}_d$ then $\hat{f} = G_{1/t} \hat{P}$ for some $\hat{P} = \sum_{d'=0}^d \hat{P}_{d'}$ with each $\hat{P}_{d'} \in \mathcal{P}_{d'}$ and $\hat{P}_d = i^d t^{-(\frac{n}{2}+d)} P$. As before $t^{-(\frac{n}{2}+d)}$ denotes the $-(n+2d)$ power of the principal square root of t .*

Proof: We use induction on d . The base case $d = 0$ is the fact that the Fourier transform of $e^{-\pi t \langle x, x \rangle}$ is $t^{-n/2} e^{-\pi \langle y, y \rangle / t}$, which we showed already. Suppose we have established the claim for $P \in \mathcal{P}_d$. By linearity and the fact that \mathcal{P}_{d+1} is spanned by its subspaces $x_j \mathcal{P}_d$, it is enough to prove the Proposition with P replaced by $x_j P$. By part (i) of Lemma 4, the Fourier transform of $G_t x_j P = x_j G_t P$ is

$$\frac{1}{2\pi i} \frac{\partial}{\partial y_j} (G_{1/t} \hat{P}) = \frac{1}{2\pi i} G_{1/t} \left(\frac{\partial \hat{P}}{\partial y_j} - \frac{2\pi}{t} y_j \hat{P} \right). \quad (22)$$

By the inductive hypothesis \hat{P} has degree d and leading part $\hat{P}_d = i^d t^{-(\frac{n}{2}+d)} P$. Therefore the right-hand side of (22) has degree $d+1$ and leading part

$$\frac{-2\pi t^{-1}}{2\pi i} \hat{P}_d = \frac{i}{t} \hat{P}_d = i^{d+1} t^{-(\frac{n}{2}+d+1)} y_j P.$$

This completes the inductive step and the proof, Q.E.D.

Proof of Theorem 1: Suppose $P \in \mathcal{P}_d^0$ and $f(x) = P(x) e^{-\pi\langle x, x \rangle t} = G_t P$. By the Corollary to Lemma 3,

$$(\Delta + \pi t(4E + 2n) + (2\pi t)^2 F)f = 0, \quad (E + 2\pi t F)f = d \cdot f. \quad (23)$$

Taking the Fourier transform and applying Lemma 4(ii), we deduce

$$(-(2\pi)^2 F - \pi t(4E + 2n) - t^2 \Delta)\hat{f} = 0, \quad -(E + n + \frac{t}{2\pi} \Delta)\hat{f} = d \cdot \hat{f}. \quad (24)$$

Eliminating $\Delta \hat{f}$, we find $d \cdot \hat{f} = (E + \frac{2\pi}{t} F)\hat{f}$; that is, \hat{f} is in the d -eigenspace of $E + 2\pi t^{-1} F$. By Proposition 3 we know $\hat{f} = G_{1/t} \hat{P}$ for some $\hat{P} \in \mathcal{P}$. By Lemma 3, then, \hat{P} is in the d -eigenspace of E ; that is, $\hat{P} \in \mathcal{P}_d$. By Proposition 3 we conclude that $\hat{P} = i^d t^{-(\frac{n}{2} + d)} P$, Q.E.D.

A natural application of weighted theta functions is to the question of equidistribution of lattice points in spherical shells. The $N_k(L)$ lattice vectors $v \in L$ on the sphere $\{x \in \mathbf{R}^n \mid \langle x, x \rangle = k\}$ yield a configuration, call it

$$S_k(L) := k^{-1/2} \{v \in L \mid \langle v, v \rangle = k\}, \quad (25)$$

of $N_k(L)$ vectors on the unit sphere $\Sigma \subset \mathbf{R}^n$. As $k \rightarrow \infty$ through the lattice norms, are the S_k asymptotically equidistributed on Σ ? Recall that the sets in a sequence $\{C_m\}_{m=1}^\infty$ of nonempty finite subsets of Σ are *asymptotically equidistributed* if, for every continuous function $\varphi : \Sigma \rightarrow \mathbf{C}$, the average of φ over C_m approaches the average of φ over Σ as $m \rightarrow \infty$:

$$\int_{x \in \Sigma} \varphi(x) d\nu_x = \lim_{m \rightarrow \infty} \frac{1}{\#C_m} \sum_{x \in C_m} \varphi(x). \quad (26)$$

Here $d\nu$ is the invariant measure on Σ such that $\int_{x \in \Sigma} d\mu_x = 1$; for instance we may define

$$\int_{x \in \Sigma} \varphi(x) d\nu_x = t^{n/2} \int_{x \in \mathbf{R}^n} e^{-\pi t \langle x, x \rangle} \varphi\left(\frac{x}{\langle x, x \rangle^{1/2}}\right) d\mu_x \quad (27)$$

for any $t > 0$. The coefficients of weighted theta functions $\Theta_{L,P}$ give us the sum in (26) when P is a harmonic polynomial. Using the decomposition (17) from Proposition 2(ii), we show that these are enough to test equidistribution:

Proposition 4 $\mathcal{P}^0|_\Sigma$ is dense in $\mathcal{C}(\Sigma)$; that is, for every continuous $\varphi : \Sigma \rightarrow \mathbf{C}$ and any $\epsilon > 0$ there exists $P \in \mathcal{P}^0$ such that

$$\forall x \in \Sigma : |P(x) - \varphi(x)| < \epsilon. \quad (28)$$

Proof: By the Stone–Weierstrass theorem there exists $P \in \mathcal{P}$ satisfying (28). It is thus enough to prove that for every $P \in \mathcal{P}$ there exists $Q \in \mathcal{P}^0$ such that $P = Q$ on Σ . Applying Proposition 2(ii) to each homogeneous part of P we write

$$P = \sum_{k=0}^{\lfloor \deg(P)/2 \rfloor} \langle x, x \rangle^k Q_k \quad (29)$$

for some $Q_k \in \mathcal{P}^0$. Since $\langle x, x \rangle = 1$ on Σ , the polynomial $Q = \sum_k Q_k \in \mathcal{P}^0$ agrees with P on Σ , Q.E.D.

Theorem 2. A sequence $\{C_m\}_{m=1}^\infty$ of nonempty finite subsets of Σ is asymptotically equidistributed if and only if

$$\lim_{m \rightarrow \infty} \frac{1}{\#C_m} \sum_{x \in C_m} P(x) = 0 \quad (30)$$

for all harmonic polynomials P of positive degree.

Proof: For the “only if” direction, assume (26) holds for all $\varphi \in \mathcal{C}(\Sigma)$, and take $\varphi = P|_\Sigma$. We claim (26) is then equivalent to (30), i.e., that $\int_{x \in \Sigma} P(x) d\nu_x = 0$. Equivalently, we claim $\int_{x \in \mathbf{R}^n} P(x) e^{-\pi t \langle x, x \rangle} d\mu_x = 0$. But the integral is the value at $y = 0$ of the Fourier transform of $P(x) e^{-\pi t \langle x, x \rangle}$. We obtained this Fourier transform in Theorem 1; it vanishes at $y = 0$ as claimed, because $P(0) = 0$ for P of positive degree.

Since both sides of (26) are linear, we have thus proved the converse implication for functions φ that are the restriction to Σ of any finite linear combination of harmonic polynomials of positive degree. We can drop the condition of positive degree, because (26) holds automatically for $\varphi = 1$: its left-hand side equals 1, and the right-hand side reduces to $\lim_{m \rightarrow \infty} 1$. Thus (30) implies (26) for all φ of the form $P|_\Sigma$ with $P \in \mathcal{P}^0$. By Proposition 4, every continuous $\varphi : \Sigma \rightarrow \mathbf{C}$ can be uniformly approximated by such $P|_\Sigma$. Hence (26) holds for all $\varphi \in \mathcal{C}(\Sigma)$ by the following standard argument. Changing φ to P moves both $\int_{x \in \Sigma} \varphi(x) d\nu_x$ and each average $(\#C_m)^{-1} \sum_{x \in C_m} \varphi(x)$ by at most ϵ . For large enough m , the average of P over C_m is within ϵ of $\int_{x \in \Sigma} P(x) d\nu_x$. Therefore the average and integral of φ are within 3ϵ of each other. Since ϵ is arbitrary, we are done. Q.E.D.

Further Remarks: The commutation relations in Lemma 1 are tantamount to an isomorphism of Lie algebras from \mathfrak{sl}_2 to the span of $\{\Delta, E + \frac{n}{2}, F\}$ that takes the standard basis $(X, H, Y) = ((0 \ 1), (\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}), (\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}))$ of \mathfrak{sl}_2 to $(\frac{1}{4\pi}\Delta, -(E + \frac{n}{2}), -\pi F)$. The decomposition $\mathcal{P}_d = \bigoplus_{k=0}^{\lfloor d/2 \rfloor} \mathcal{P}_d^k$ in Proposition 2(ii) then says in effect that $\mathcal{P} = \bigoplus_{d=0}^\infty \mathcal{P}_d^0 \otimes V_{\frac{n}{2}+d}$, where for any real $m > 0$ we write V_m for the infinite-dimensional irreducible representation of \mathfrak{sl}_2 with basis $\{Y^k v\}_{k=0}^\infty$ where $Xv = 0$ and $Hv = -mv$.

Moreover, we noted already that $G_t = \exp(-\pi t F)$, and it is known that the Fourier transform is $e^{-\pi i n/4} \exp \frac{\pi i}{2} (\pi F - \frac{1}{4\pi} \Delta)$.⁴ Thus Lemmas 3 and 4, which give the action on Δ, E, F of conjugation by G_t and the Fourier transform, correspond to the action on \mathfrak{sl}_2 of conjugation by the elements $e^{tY} = (\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix})$ and $e^{-\pi i (X+Y)/2} = -(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix})$ of SL_2 .

In Proposition 3, we obtain \widehat{P} by applying to P the operators G_t , then the Fourier transform, then

⁴This has a memorable physical interpretation: running the Schrödinger equation on a quantum harmonic oscillator for $1/4$ of its classical period applies a multiple of the Fourier transform to the wave function. The distribution of factors of π in this formula is the reason we chose the homomorphism from \mathfrak{sl}_2 that maps $(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix})$ and $(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix})$ to $\frac{1}{4\pi}\Delta$ and $-\pi F$, rather than $\frac{1}{2}\Delta$ and $-\frac{1}{2}F$ which seems more natural at first.

Alternatively, we could use Lemma 4 to identify the Fourier transform with conjugation by $\pm i(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$.

$G_{-1/t}$; up to the constant factor $e^{-\pi in/4}$, this corresponds to the product

$$\begin{pmatrix} 1 & 0 \\ -1/t & 1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = -i \begin{pmatrix} t & 1 \\ 0 & -1/t \end{pmatrix}$$

in SL_2 , which can be written as

$$-i \begin{pmatrix} t & 1 \\ 0 & -1/t \end{pmatrix} \begin{pmatrix} 1 & 1/t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t/i & 0 \\ 0 & (it)^{-1} \end{pmatrix} \begin{pmatrix} 1 & i/t \\ 0 & 1 \end{pmatrix}.$$

Now if $P \in \mathcal{P}_d^0$ then P is fixed by the one-parameter subgroup $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ of SL_2 generated by X . Thus \widehat{P} is a multiple of the image of P under a diagonal matrix in SL_2 . Such diagonal matrices constitute the one-parameter group generated by H . Since P is an eigenvector of H , we deduce that \widehat{P} is proportional to P . More precisely, since $HP = -(\frac{n}{2} + d)P$ we have $\mathrm{diag}(e^\beta, e^{-\beta})P = e^{\beta H}P = e^{-\beta(\frac{n}{2} + d)}P$; taking $e^\beta = t/i$ (with $\mathrm{Im}(\beta) = -\pi/2$),⁵ and restoring the factor $e^{-\pi in/4}$, we recover

$$\widehat{P} = e^{-\pi in/4} (t/i)^{-(\frac{n}{2} + d)} P = i^d t^{-(\frac{n}{2} + d)} P,$$

which is Theorem 1.

The differential operators $\partial^2/\partial x_j \partial x_k$, $x_j \partial/\partial x_k + \frac{1}{2} \delta_{jk}$, and $x_j x_k$ generate a Lie algebra isomorphic with \mathfrak{sp}_{2n} , which contains the span of $\Delta, \mathbf{E} + \frac{n}{2}, \mathbf{F}$ as the subalgebra invariant under the orthogonal group of \mathbf{R}^n . Multiplying $\partial^2/\partial x_j \partial x_k$ and $x_j x_k$ by i (to make all the generators skew-Hermitian) yields infinitesimal generators of the Lie algebra of Weil's projective representation⁶ of $\mathrm{Sp}_{2n}(\mathbf{R})$ on $L^2(\mathbf{R}^n)$.

⁵We must specify the path because in general a representation of \mathfrak{sl}_2 lifts to a representation not of SL_2 but of its universal cover. A more honest treatment would either carefully check that we are working in a contractible patch of $\mathrm{SL}_2(\mathbf{C})$, or compute the constant factor i^d of (5) in some other way.

⁶André Weil: Sur certaines groupes d'opérateurs unitaires, *Acta Math.* **111**, 143–211 (1964).