Math 272y: Rational Lattices and their Theta Functions

7 October 2019: Gluing

We have seen that if L_1 is a self-dual sublattice of an integral lattice L then L_1 is a "direct summand": L decomposes as the direct sum $L_1 \oplus L_2$ where L_2 is the orthogonal complement of L_1. In general if L_1 is nondegenerate then $L_1 \oplus L_2$ is a finite-index sublattice of L. To recover L we need some additional information, which lets us "glue" L_1 to L_2. If L is self-dual, we shall obtain a particularly nice description: if L_1 is saturated in L then the discriminant groups of L_1 and L_2 are related by a canonical isomorphic that multiplies the discriminant form by -1, and conversely any such "anti-isomorphism" between L_1^*/L_1 and L_2^*/L_2 yields a self-dual lattice $L \supseteq L_1 \oplus L_2$. This lets us use our work on self-dual lattices to describe and sometimes to classify lattices of discriminant Δ for some $\Delta > 1$ even before delving into the modularity of theta functions of such lattices.

We begin with some easy observations on the ranks of even lattices whose discriminant is odd or singly even.[1] For now we do not require L to be definite.

Proposition 1. *Suppose L is an even lattice of rank n. If $\operatorname{disc} L$ is odd then n is even, and then $\operatorname{disc} L \equiv (-1)^{n/2} \bmod 4$. If $\operatorname{disc} L \equiv 2 \bmod 4$ then n is odd.*

Note that there can be no constraint on the rank of an even lattice of discriminant 4, because D_n provides a rank-n example for every n.

We prove the Proposition by repeated application of the following lemma, in which L is not required to be even, and is allowed to be degenerate (though L_1 is not).

Lemma 1. *Suppose L is an integral lattice and L_1 a nondegenerate sublattice. Let π be the orthogonal projection $L \to L_1 \otimes \mathbf{Q}$, and let $L_2 = \ker \pi$ be the orthogonal complement of L_1 in L. Then $\pi(L) \subseteq L_1^*$, and the kernel of the homomorphism $v \mapsto \pi(v) \bmod L_1$ is $L_1 \oplus L_2$. In particular, the index $[L : L_1 \oplus L_2]$ is a factor of $\operatorname{disc} L_1$.*

Proof of Lemma: Let v be any vector in L, and $v_1 = \pi(v)$. Then $\langle v_1, w\rangle = \langle v, w\rangle \in \mathbf{Z}$ for all $w \in L_1$, so $v_1 \in L_1^*$. If $v_1 \in L_1$ then $v - v_1 \in L_2$, so $v \in L_1 \oplus L_2$, and conversely; hence $L_1 \oplus L_2$ is the kernel of the homomorphism $L \to L_1^*/L_1$ sending any v to the coset of $\pi(v)$, as claimed. This gives an injection from $L/(L_1 \oplus L_2)$ to a subgroup of L_1^*/L_1, so in particular $[L : L_1 \oplus L_2]$ is a factor of $\left|L_1^*/L_1\right| = \operatorname{disc} L_1$. $\square$

Now to prove Proposition 1. If $n = 1$ then $L = \mathbf{Z}v$ for some v and then $\operatorname{disc} L = \langle v, v\rangle$ is even. If $n = 2$, let A be a Gram matrix with entries a_{ij}, so $\operatorname{disc} L = a_{11}a_{22} - a_{12}^2$. The first term is a multiple of 4, and $a_{12}^2 \equiv 0$ or $1 \bmod 4$, so $\operatorname{disc} L$ is either 0 or $-1 \bmod 4$ as claimed. For larger n, we argue inductively, using Lemma 1 to reduce rank n to rank $n-2$. Choose some basis $v_1, \ldots, v_n$ for L. Since $\operatorname{disc} L$ is not a multiple of 2^n, one of the entries $\langle v_i, v_j\rangle$ of the Gram matrix must be

[1] An even integer n is said to be "singly even" or "doubly even" according as $n \equiv 2$ or $n \equiv 0 \bmod 4$; that is, a singly even n has valuation $v_2(n) = 1$, and a doubly even n has $v_2(n) \geq 2$ (*not* necessarily $v_2(n) = 2$). This is a reasonably common term; much rarer is "triply even" for $8|n$, let alone "quadruply even" and beyond.

odd, and it is not a diagonal entry because L is even. Let L_1 be the sublattice generated by v_i and v_j, and L_2 its orthogonal complement. Then $\operatorname{disc} L_1 \equiv -1 \bmod 4$, and

$$\operatorname{disc} L = \operatorname{disc}(L_1 \oplus L_2) \,/\, [L : L_1 \oplus L_2]^2 = \operatorname{disc} L_1 \operatorname{disc} L_2 \,/\, [L : L_1 \oplus L_2]^2.$$

By Lemma 1, $[L : L_1 \oplus L_2]$ is a factor of $\operatorname{disc} L_1$, so $[L : L_1 \oplus L_2]$ is odd and $[L : L_1 \oplus L_2]^2 \equiv 1 \bmod 4$. Hence $\operatorname{disc} L \equiv \operatorname{disc} L_1 \operatorname{disc} L_2 \bmod 4$. But $\operatorname{disc} L_1 \equiv -1 \bmod 4$, and L_2 has rank $n-2$. Thus if n is odd, then so is $n-2$, so by the inductive hypothesis $\operatorname{disc} L_2$ is even, whence the same is true of $\operatorname{disc} L$. If n is even and $\operatorname{disc} L$ is odd, then the same is true of $\operatorname{disc} L_2$, which is then $\equiv (-1)^{(n/2)-2} \bmod 4$ by the inductive hypothesis, whence $\operatorname{disc} L \equiv -(-1)^{(n/2)-2} = (-1)^{n/2}$. In either case this completes the induction step and the proof. □

The result that $\operatorname{disc} L$ is even if n is odd can also be proved by observing that $A \bmod 2$ is an alternating matrix (antisymmetric with all diagonal entries zero): an alternating matrix of odd order over any field is singular, so $\operatorname{disc} L = \det A \equiv 0 \bmod 2$.

If L is also positive-definite with $\operatorname{disc} L = 1$ then Proposition 1 gives $4|n$, but we know that in fact $8|n$. We shall use gluing to give similar refinements for other discriminants. Before setting out the general framework we illustrate with the case of discriminant 2.

Proposition 2. *Suppose L is a positive-definite even lattice of rank n and discriminant 2, and let v be any vector with $v \in L^*$ but $v \notin L$. Then either $n \equiv 1 \bmod 8$ and $\langle v, v\rangle \equiv \frac{1}{2} \bmod 2$, or $n \equiv -1 \bmod 8$ and $\langle v, v\rangle \equiv -\frac{1}{2} \bmod 2$.*

(Note that this is consistent with what we know about the root lattices A_1 and E_7.)

Proof: We have $2v \in L$ because $[L^* : L] = 2$. Let $a = \langle v, v\rangle$. We first show that $a \in \frac{1}{2}\mathbf{Z}$ but $a \notin \mathbf{Z}$. This much is easy even under a rather weaker hypothesis:

Lemma 2. *Suppose L is an even lattice with $\operatorname{disc} L \equiv 2 \bmod 4$, and let $v \in L^*$ with $2v \in L^*$ but $v \notin L$. Then $2\langle v, v\rangle \in \mathbf{Z}$ but $\langle v, v\rangle \notin \mathbf{Z}$.*

Proof. Since $2v \in L$ we know that $4\langle v, v\rangle = \langle 2v, 2v\rangle \in 2\mathbf{Z}$; hence $\langle v, v\rangle \in \frac{1}{2}\mathbf{Z}$. Now suppose for the sake of contradiction that $\langle v, v\rangle \in \mathbf{Z}$, and consider the lattice L' generated by L and v. Then L' is integral, so $\operatorname{disc} L'$ is an integer. But then $[L' : L] = 2$, so $\operatorname{disc} L = [L' : L]^2 \operatorname{disc} L' = 4 \operatorname{disc} L' \equiv 0 \bmod 4$, contradicting $\operatorname{disc} L \equiv 2 \bmod 4$. Therefore $\langle v, v\rangle \notin \mathbf{Z}$, as claimed. □

In the setting of Proposition 2, it remains to show that if $\langle v, v\rangle \equiv \frac{1}{2} \bmod 2$ then $n \equiv 1 \bmod 8$, while if $\langle v, v\rangle \equiv -\frac{1}{2} \bmod 2$ then $n \equiv -1 \bmod 8$. In each case we do this by choosing some positive-definite even lattice M of discriminant 2 and "gluing L to M" by constructing an even unimodular lattice L' that contains $L \oplus M$ with index 2. Then the rank of L' must be a multiple of 8, from which we deduce the claimed congruence on n.

Assume first that if $\langle v, v\rangle \equiv -\frac{1}{2} \bmod 2$, and take $M = A_1$. Then $L \oplus M$ has rank $n + 1$ and discriminant 4. Let w be a generator of A_1^*, so $\langle w, w\rangle = 1/2$; and let L' be the lattice generated by $L \oplus M$ and (v, w). Then $[L' : (L \oplus M)] = 2$, so $\operatorname{disc} L' = 4/2^2 = 1$. We claim L' is even. We first check that it is integral: $L \oplus M$ is integral, and the new vector (v, w) has integral inner product with any vector of $L \oplus M$ (since $(v, w) \in (L \oplus M)^*$) and with itself. Then, to check that L' is even, we need only check that it is generated by vectors of even norm; this is true of $L \oplus M$, and the new vector's norm is congruent mod 2 to $(-\frac{1}{2}) + \frac{1}{2} = 0$. So L' is positive-definite, self-dual, and even, whence its rank $n+1$ is a multiple of 8. This proves that $n \equiv -1 \bmod 8$ in this case.

If instead $\langle v, v\rangle \equiv +\frac{1}{2} \bmod 2$, we take $M = E_7$, and choose any $w \in E_7^*$ that is not in E_7. Again we let L' be the lattice generated by $L \oplus M$ and (v, w). Since $\operatorname{disc} E_7 = 2$ and $\langle w, w\rangle \equiv \frac{3}{2} \bmod 2$, we conclude again that L' is even, this time with (v, w) having norm congruent mod 2 to $\frac{1}{2} + (-\frac{1}{2}) = 0$. Here L' has rank $n + 7$; since this must be a multiple of 8, we deduce that $n \equiv +1 \bmod 8$ in this case. This completes the proof. □

Corollary. *The minimal discriminant of a positive-definite even lattice of rank* n *depends on* $n \bmod 8$ *as follows:*

n	$8m$	$8m+1$	$8m+2$	$8m+3$	$8m+4$	$8m+5$	$8m+6$	$8m+7$
$\min(\operatorname{disc} L)$	1	2	3	4	4	4	3	2
example	E_8^m	$E_8^m \oplus A_1$	$E_8^m \oplus A_2$	$E_8^m \oplus A_3$	$E_8^m \oplus D_4$	$E_8^m \oplus D_5$	$E_8^m \oplus E_6$	$E_8^m \oplus E_7$

Proof: The examples in the table show that the claimed minimum is attained in each case, so it remains to show that the discriminant can be no smaller. We already know that $\operatorname{disc} L = 1$ occurs only for $n = 8m$. Proposition 2 shows that $\operatorname{disc} L = 2$ implies $n = 8m + 1$ or $n = 8m + 7$, and Proposition 1 shows that $\operatorname{disc} L = 3$ implies $n = 8m + 2$ or $8m + 6$. □

The proof of Proposition 2 also tells us that L is the orthogonal complement of M in L'. Thus classifying even lattices of discriminant 2 in $\mathbf{R}^n$ comes down to classifying even self-dual lattices L' in $\mathbf{R}^{n+1}$ or $\mathbf{R}^{n+7}$ together with a sublattice isomorphic to M. Since we know all such L' of rank 8 and 16, we can do this for each $n < 16$; once we describe all the Niemeier lattices we'll be able to extend this to $n = 17$ and $n = 23$ as well. The bimodular even lattices in $\mathbf{R}^{25}$ are classified in Borcherds' thesis as orthogonal complements of vectors of norm -2 in the unique even unimodular lattice in hyperbolic space $\mathbf{R}^{25+1}$. For $n = 31$ and beyond, a full classification is not feasible with current methods.

For now we treat $n < 16$. For $n = 1$ we already know that $L \cong A_1$. For $n = 7$ we've seen that $L' \cong E_8$, so L is the orthogonal complement of a root in E_8; all the E_8 roots are equivalent under $W(E_0)$, so L is unique up to isomorphism, whence $L \cong E_7$. For $n = 9$ we need a copy of E_7 into E_8^2 or D_{16}^+. Since E_7 is a simple root lattice, an embedding into any positive-definite integral lattice L' must put E_7 in one of the simple components of $R(L')$. We noted already that no $\mathbf{Z}^N$ accommodates one of the E_n lattices; since $D_{16} \subset \mathbf{Z}^{16}$, there is no copy of E_7 in D_{16}^+. So we must put E_7 in one of the E_8 factors of E_8^2, which makes $L \cong A_1 \oplus E_8$. Finally for $n = 15$ there are two choices. For either $L' = E_8^2$ or $L' = D_{16}^+$, all roots are equivalent under $\operatorname{Aut}(L')$, and thus produce isomorphic L. For $L' = E_8^2$ this makes $L \cong E_7 \oplus E_8$. For $L' = D_{16}^+$, the orthogonal complement in D_{16} of any root is $A_1 \oplus D_{14}$, so we have $L' \cong (A_1 \oplus D_{14})^+$, a lattice containing $A_1 D_{14}$ with index 2 and generated by $A_1 D_{14}$ and (v, w) where v is any half-lattice vector and $w \in D_{14}^*$ has norm $7/2$. This last construction is again an example of "gluing" (of A_1 to D_{14}).

Positive-definite even lattices of discriminant 3 can be analyzed similarly, using A_2 or E_6 as the complementary lattice M instead of A_1 or E_7. This does not give new restrictions on n but does show that the dual vectors of L^* that are not in L have norm congruent mod 2 to $2/3$ or $4/3$ according as $n \equiv 2$ or $6 \bmod 8$. We can also recover the classification even such lattices in rank < 16, finding that they are unique for $n = 2, 6, 10$ but there are two non-isomorphic lattices for $n = 14$ (namely $E_6 \oplus E_8$ and the complement of A_2 in D_{16}^+).

For further applications of this technique to classifying positive-definite lattices of small discriminant and rank, whether even or not, see Chapter 15, section 10.2–3 (pages 399–402) of SPLAG, where this technique is called the "Kneser gluing method" (referring to Kneser's paper "Klassenzahlen definiter quadratischer Formen", pages 241–250 in *Archiv Math.* **8** (1957)).

We do not try to give the most general result of this kind. Such a result would almost amount to the classification of all genera of quadratic forms, which is known but somewhat complicated. Fortunately will not need such generality, and any specific case is more easily handled by a gluing argument.

We do give a theorem that describes gluing any two lattices L_1, L_2 to a self-dual lattice. Suppose L is a self-dual lattice, and let L_1 be a nondegenerate sublattice. (If L is positive-definite then L_1 is automatically nondegenerate, but we allow indefinite L too.) Let L_2 be the orthogonal complement:

$$L_2 = \{v_2 \in L : \forall v_1 \in L_1, \langle v_1, v_2 \rangle = 0\}.$$

Then L_2 is *saturated* in L; that is, $L_2 = (L_2 \otimes \mathbf{Q}) \cap L$ (equivalently: L/L_2 is a free abelian group). Moreover, the orthogonal complement of L_2 is the *saturation* $(L_1 \otimes \mathbf{Q}) \cap L$ of L_1 in L. We thus assume that L_1 is saturated, so L_1 and L_2 are each other's orthogonal complements in L. We shall show that $\operatorname{disc} L_1 = \pm \operatorname{disc} L_2$, and indeed that the discriminant groups L_1^*/L_1 and L_2^*/L_2 are canonically isomorphic. More precisely:[2]

Theorem. *Let L be a self-dual lattice, L_1 a nondegenerate saturated sublattice of L, and L_2 the orthogonal complement of L_1. Then the orthogonal projections $\pi_j : L \to L_j \otimes \mathbf{Q}$ $(j = 1, 2)$ map L onto L_j^*, with $\pi_1^{-1}(L_1) = \pi_2^{-1}(L_2) = L_1 \oplus L_2$. The resulting isomorphisms from $L/(L_1 \oplus L_2)$ to L_1^*/L_1 and L_2^*/L_2 yield an isomorphism $g : L_1^*/L_1 \to L_2^*/L_2$ such that $(gc, gc') = -(c, c')$ for any classes $c, c' \in L_1^*/L_1$; if L is even then moreover $q(gc) = -q(c)$ for all $c \in L_1^*/L_1$. Conversely, if L_1 and L_2 are nondegenerate integral lattices, and $g : L_1^*/L_1 \to L_2^*/L_2$ is any isomorphism such that $(gc, gc') = -(c, c')$ for all $c, c' \in L_1^*/L_1$ then*

$$L := \{(v_1, v_2) \in L_1^* \times L_2^* : v_2 + L_2 = g(v_1 + L_1)\} \tag{1}$$

is self-dual; if moreover L_1 and L_2 are even, and $q(gc) = -q(c)$ for all $c \in L_1^*/L_1$, then L is even.

Here $(\cdot, \cdot)$ and $q(\cdot)$ are the canonical $\mathbf{Q}/\mathbf{Z}$- and $\mathbf{Q}/2\mathbf{Z}$-valued pairing and quadratic form on the discriminant group. For example, if $L = \mathbf{Z}^{n+1}$ and $L_1 = \mathbf{Z} \sum_{j=1}^{n+1} e_j$ then $L_2 = A_n$. Since $L_1 \cong \mathbf{Z}\langle n+1 \rangle$, the discriminant group L_1^*/L_1 is cyclic of order $n+1$, generated by c_1 with $(c_1, c_1) = 1/(n+1) \in \mathbf{Q}/\mathbf{Z}$. Thus the Theorem asserts that A_n^*/A_n is also cyclic of order $n+1$,

[2] The isomorphism between L_1^*/L_1 and L_2^*/L_2 is given (without proof) by Conway and Sloane as Theorem 1 on page 100.

with a generator c_2 such that $(c_2, c_2) = -1/(n+1) \in \mathbf{Q}/\mathbf{Z}$. This agrees with our computation from last week (where the c_j are the images of a unit vector of $\mathbf{Z}^{n+1}$ under the maps from $\mathbf{Z}^{n+1}$ to L_j^*/L_j).

Proof: For any $v \in L$ and $w_1 \in L_1$ we have $\langle \pi_1(v), w_1 \rangle = \langle v, w \rangle \in \mathbf{Z}$; this together with $\pi_1(v) \in L_1 \otimes \mathbf{Q}$ implies that $\pi_1(v) \in L^*$. Conversely, if $v^* \in L_1^*$ then $w \mapsto \langle v^*, w \rangle$ is a homomorphism $L_1 \to \mathbf{Z}$; because L_1 was assumed to be saturated in L, this homomorphism extends to a homomorphrism $L \to \mathbf{Z}$, which is represented by some $v \in L$ because L is self-dual. This proves that $\pi_1 : L \twoheadrightarrow L_1^*$, and the same argument gives $\pi_2 : L \twoheadrightarrow L_2^*$. Moreover, if $\pi_1(v) \in L_1$ then $\pi_2(v) = v - \pi_1(v) \in L$, whence $\pi_2(v) \in L \cap L_2^* = L_2$ because L_2 is saturated in L. This proves that $\pi_1^{-1}(L_1) = L_1 \oplus L_2$, and again the same argument gives $\pi_2^{-1}(L_2) = L_1 \oplus L_2$.

Now the map taking any $v \in L$ to $\pi_1(v) \bmod L_1$ has image L_1^*/L_1 and kernel $L_1 \oplus L_2$, so gives an isomorphism from $L/(L_1 \oplus L_2)$ to L_1^*/L_1; in the same way $\pi_2 \bmod L_2$ gives an isomorphism from $L/(L_1 \oplus L_2)$ to L_2^*/L_2. This give our isomorphism g, characterized by $g(\pi_1(v)+L_1) = \pi_2(v)+L_2$ for all $v \in L$. For any $c, c' \in L_1^*/L_1$ we can find preimages $x, y \in L$, and then the integer $\langle x, y \rangle$ is $\langle \pi_1 x, \pi_1 y \rangle + \langle \pi_2 x, \pi_2 y \rangle$, in which the first term is $(c, c') \bmod \mathbf{Z}$ and the second is $(gc, gc') \bmod \mathbf{Z}$. Therefore $(gc, gc') = -(c, c')$. If L is even, then for any $c, \in L_1^*/L_1$ we can find a preimage $x \in L$, and then the even integer $\langle x, y \rangle$ is $\langle \pi_1 x, \pi_1 x \rangle + \langle \pi_2 x, \pi_2 x \rangle$, in which the first term is $q(c) \bmod 2\mathbf{Z}$ and the second is $q(gc) \bmod 2\mathbf{Z}$. Therefore $q(gc) = -q(c)$.

The converse result (properties of the lattice (1) constructed from g) is an exercise. $\square$

The notation g is meant to suggest "gluing map"; it specifies the "gluing" of L_1 to L_2 to recover L in (1). In this context Conway and Sloane use the name "glue group" for the discriminant group L^*/L of a nondegenerate integral lattice L.