Math 272y: Rational Lattices and their Theta Functions

25 November 2019: Lattices of level 3 and their theta functions

We briefly describe level-3 lattices and their theta functions, highlighting some differences and new phenomena compared with what we saw for level 2.

Let $L \subset \mathbf{R}^n$ be a lattice of level 3, that is, an even lattice such that $L' := (L^*)\langle 3 \rangle$ is also even. (As with level 2, we may allow either L or L' to be self-dual, and thus of level 1, though we have dealt with such lattices already.) Since 3 is odd and squarefree, it is enough to assume that $3L^* \subseteq L$: then if $v \in L^*$ then $3v \in L$ so $\langle v, v \rangle = \frac{1}{3}\langle 3v, v \rangle \in \frac{1}{3}\mathbf{Z}$, and also $\langle v, v \rangle = \frac{1}{9}\langle 3v, 3v \rangle \in \frac{1}{9}2\mathbf{Z}$, so $\langle v, v \rangle \in \frac{2}{3}\mathbf{Z}$; hence $(L^*)\langle 3 \rangle$ is an even lattices, as claimed.

Necessarily n is even because $\operatorname{disc} L$ is odd. This time any even n may arise; we have seen already the example of A_2, which yields a level-3 lattice $A_2^{n/2} \subset \mathbf{R}^n$ for all even n.

Next we describe the discriminant forms on the $(\mathbf{Z}/3\mathbf{Z})$-vector spaces

$$\mathrm{D}(L) = L^*/L, \quad \mathrm{D}(L') = L'^*/L',$$

each of which carries a nondegenerate quadratic form $Q : [v] \mapsto (3\langle v, v \rangle) \bmod 3$.

We recall the structure of quadratic forms over finite fields of odd characteristic. For a finite field k of odd order q, it is known that for each $r > 0$ there are two isomorphism classes of nondegenerate quadratic forms on r-dimensional vector spaces V/k, classified by the discriminant which is well-defined in k^*/k^{*2}. For r odd, we can scale a form in one class to the other, so there is only one orthogonal group. For r even, the isomorphism class is scaling invariant, and the two classes are also distinguished by their maximal isotropic spaces: if the discriminant is $(-1)^{r/2}$ times a square then V has isotropic subspaces of dimension $r/2$; otherwise the largest isotropic subspaces have dimension $(r/2)-1$. (Ultimately the factor $(-1)^{r/2}$ arises because for $r = 2$ the split quadratic form $Q(x_1, x_2) = x_1^2 - x_2^2$ has discriminant -1.) We use Q_+ for a quadratic form that has an $(r/2)$-dimensional isotropic subspace, and Q_- for a form that does not; as we saw in characteristic 2, the orthogonal sum of two $Q+$ forms or of two Q_- forms is of type Q_+, while the sum of a Q_+ and a Q_- is of type Q_-.

Now for $q = 3$ the only nonzero square is 1, so quadratic forms have a well-defined discriminant in $\{\pm 1\}$. The quadratic forms on $\mathrm{D}(A_2)$ and $\mathrm{D}(E_6)$ have discriminants -1 and $+1$ respectively, while the form on $\mathrm{D}(A_2^2)$ has type Q_-. We deduce:

Proposition. *Let $L \subset \mathbf{R}^n$ be a lattice of level 3 and discriminant 3^r. Then $r \equiv n/2 \bmod 2$. If $n \equiv 2 \bmod 4$ then the quadratic form on $\mathrm{D}(L)$ has discriminant $(-1)^{(n/2)+r}$. If $4|n$ then the quadratic form on $\mathrm{D}(L)$ is of type Q_+ or Q_- according as n is 0 or 4 $\bmod 8$.*

Proof: If r is even and the form has type Q_+ then the preimage of a maximal isotropic is an even unimodular lattice of rank n, so $8|n$. If the type is Q_-, the form on the rank $n + 4$ lattice $L \oplus A_2^2$ has type Q_+, so $8 \mid n + 4$ and $n \equiv 4 \bmod 8$. Conversely, if r is odd then $L \oplus A_2$ has rank $n + 2$ and level 3 with and discriminant 3^{r+1}, so $4 \mid n + 2$ and $n \equiv 2 \bmod 4$. The discriminant is then deduced from the type of the quadratic form on $\mathrm{D}(L \oplus A_2)$. $\square$

Now to the modular forms.

We have seen that the translation $T : z \mapsto z + 1$ and the involution $w_3 : z \mapsto -1/(3z)$ generate a hyperbolic triangle group, with a cusp at $i\infty$ (fixed by T) and elliptic points at $i/\sqrt{3}$ (fixed by w_3) and $(i/\sqrt{3} - 1)/2$ (fixed by $w_3 T$, which has order 6). The index-2 subgroup $\Gamma_1(3)$ is also a triangle group, generated by T and $w_3 T w_3 : z \mapsto z/(1 - 3z)$, with cusps at 0 (fixed by $w_3 T w_3$) and $i\infty$, and an elliptic point at $(i/\sqrt{3} - 1)/2$ fixed by $(w_3 T)^2 : z \mapsto -(z + 1)/(3z + 2)$. For $N = 3$ the congruence subgroup $\Gamma_1(N)$ no longer contains $-I_2$, but it still contains either g or $-g$ for every $g \in \Gamma_0(N)$. (This is also the case for $N = 4$ and $N = 6$, and for no larger N.) Thus if a holomorphic function $f : \mathcal{H} \to \mathbf{C}$ is weakly modular of some weight $k \in \mathbf{Z}$ for $\Gamma_1(3)$, then f is automatically modular of level 3, with trivial Nebentypus if k is even, and Nebentypus χ_3 (the nontrivial Dirichlet character mod 3) if k is odd.

For $k \in \mathbf{Z}$, let $M_k(\Gamma_1(3))$ be the vector space of modular forms of weight k for $\Gamma_1(3)$, and $S_k(\Gamma_1(3))$ its subspace of cusp forms. The involution w_3 acts on these spaces by

$$(w_3 f)(z) = (3^{1/2} z/i)^{-k} f(-1/3z); \tag{1}$$

as we did for w_2, we choose the factor $(3^{1/2} z/i)^{-k}$ so that it is positive on the imaginary axis, so in particular $w_3 \theta_L = \theta_L$ if L is a level-3 lattice such as A_2 for which $L \cong L'$.

Theorem. *For any integer k,*
i) $M_k(\Gamma_1(3))$ has dimension $\max(\lfloor (k + 3)/3 \rfloor, 0)$;
ii) $S_k(\Gamma_1(3))$ has dimension $\dim M_{k-6}(\Gamma_1(3)) = \max(\lfloor (k + 3)/3 \rfloor - 2, 0)$;
iii) If $M_k(\Gamma_1(3))$ or $\dim S_k(\Gamma_1(3))$ has dimension d then the w_3-invariant subspace has dimension $d/2$ or $(d + 1)/2$ according as d is even or odd.

That is:

k	< 0	0	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
$\dim M_k(\Gamma_1(3))$	0	1	1	1	2	2	2	3	3	3	4	4	4	5	$\cdots$
$\dim M_k(\Gamma_1(3))^{w_3}$	0	1	1	1	1	1	1	2	2	2	2	2	2	3	$\cdots$
$\dim S_k(\Gamma_1(3))$	0	0	0	0	0	0	0	1	1	1	2	2	2	3	$\cdots$
$\dim S_k(\Gamma_1(3))^{w_3}$	0	0	0	0	0	0	0	1	1	1	1	1	1	2	$\cdots$

The proof starts as with the parallel theorems for $\Gamma_1(2)$ (and $\Gamma(1)$). We first show that all nonzero $f \in M_k(\Gamma_1(3))$ have the same number of zeros in a fundamental domain $\mathcal{F}_3$ for $\Gamma_1(3)$, counted with multiplicity. Here the final answer is $k/3$, coming from a contour integral over the third-circle $\{z \in \mathcal{H} : |z| = 1/\sqrt{3}, \ |\mathrm{Re}(z)| \leq 1/2\}$, and consistent with the count of $k/12$ for a $\Gamma(1)$ form because $\Gamma_1(3)$ has index 4 in $\mathrm{PSL}_2(\mathbf{Z})$. As before (*mutatis mutandis*), the cusp at $i\infty$ is counted as for Γ_1 (a nonzero multiple of $q^n + O(q^{n+1})$ has valuation n); an n-th order zero at an elliptic point in the orbit of $(i/\sqrt{3} - 1)/2$ is counted with multiplicity $n/3$; and at $z = 0$ the valuation of f is the valuation of $w_3 f$ at $z = i\infty$.

This lets as show that for each k the dimensions of $M_k(\Gamma_1(3))$ and $S_k(\Gamma_1(3))$ are no larger than claimed, so it is enough to produce enough modular forms. For $k = 1$, we use θ_{A_2}; this also gives a generator $\theta_{A_2}^2$ of $M_2(\Gamma_1(3))$, and for $M_3(\Gamma_1(3))$ we use $\theta_{A_2}^3$ and θ_{E_6}. We can then construct a weight-6 cusp

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form $\Delta_{(3)} = q + O(q^2)$ as a product of two nonzero linear combinations of $\theta_{A_2}^3$ and θ_{E_6}, one vanishing at $z = i\infty$ and the other at $z = 0$. Now for each k the weight-k monomials in θ_{A_2} and θ_{E_6} constitute a basis for $M_k(\Gamma_1(3))$, and their products with $\Delta_{(3)}$ constitute a basis for $S_{k+6}(\Gamma_1(3))$. This also yields the dimensions of the w_3-invariant subspaces, once we check that θ_{A_2} and $\Delta_{(3)}$ are invariant under w_3. These two forms generate $M_1(\Gamma_1(3))$ and $S_6(\Gamma_1(3))$ respectively, so must be in either the $+1$ or -1 eigenspace for w_3, and if either of these forms were in the -1 eigenspace it would vanish at $i/\sqrt{3}$, which is not possible by the zero-counting formula.

By comparing zero multiplicities and leading terms, we see that the weight-24 form $\Delta_{(3)}^4$ equals $\Delta(z)\Delta(3z)$, whence

$$\Delta_{(3)}(z) = \bigl(\eta(z)\eta(3z)\bigr)^6 = q \prod_{m=1}^{\infty} \bigl((1 - q^m)(1 - q^{3m})\bigr)^6, \tag{2}$$

which expands to

$$\Delta_{(3)}(z) = q - 6q^2 + 9q^3 + 4q^4 + 6q^5 - 54q^6 - 40q^7 \cdots . \tag{3}$$

Likewise the weight-3 forms vanishing at a cusp are

$$\frac{1}{54}(\theta_{E_6} - \theta_{A_2}^3) = \eta(q^3)^9/\eta(q)^3 = q + 3q^2 + 9q^3 + 13q^4 + 24q^5 + 27q^6 + 50q^7 + \cdots ,$$

$$\frac{1}{2}(3\theta_{E_6'} - \theta_{A_2}^3) = \eta(q)^9/\eta(q^3)^3 = 1 - 9q + 27q^2 - 9q^3 - 117q^4 + 216q^5 + 27q^6 \cdots . \tag{4}$$

If disc $L = 3^{n/2}$ then L and L' have the same discriminant, and it may happen that $L \cong L'$, as happens for $L = A_2$. For such lattices, θ_L is invariant under w_3, so is a polynomial in θ_{A_2} and $\Delta_{(3)}$. As before, we deduce a lower bound on $N_{\min}(L)$, which is satisfied if and only if θ_L is extremal; here this bound is $2\lfloor n/12 \rfloor + 2$, and once more the extremal modular form has positive $q^{\lfloor n/12 \rfloor + 1}$ coefficient because the product formula (2) shows that the power series for $1/\Delta_{(3)}$ has positive coefficients. Extremal lattices of level 3 include at least two record sphere packings: A_2 for $n = 2$, and the *Coxeter-Todd lattice* K_{12}, which can be constructed as a 3-neighbor of A_2^6: write A_2 as $\{(x, y, z) \in \mathbf{Z}^3 : x + y + z = 0\}$; form a lattice $(A_2^6)^{+3}$ generated by A_2^6 and the dual vector (v, v, v, v, v, v) where $v = (1/3, 1/3, -2/3) \in A_2^*$; then K_{12} is the index-3 sublattice $\{(\vec{x}, \vec{y}, \vec{z}) \in A_2^6 : \sum_{j=1}^{6} x_j - y_j \equiv 0 \bmod 3\}$. This lattice has theta series

$$\theta_{K_{12}} = \theta_{A_2}^6 - 36\Delta_{(3)} = 1 + 756q^2 + 4032q^3 + 20412q^4 + 60480q^5 + \cdots . \tag{5}$$

which can be used to show that K_{12} is the only extremal lattice of rank 12 and level 3.

The shells of an extremal lattice of rank n constitute spherical 5-designs if $n \equiv 0$ or $2 \bmod 12$, 3-designs if $n \equiv 4$ or $6 \bmod 12$, and 1-designs in the remaining cases $n \equiv 8$ or $10 \bmod 12$. For example, the regular hexagon is the first shell of A_2 and is a circular 5-design, and each shell of K_{12} is a spherical 5-design. For a variation on this theme, the shells of E_6 are also 5-designs: since E_6' has no roots, any weighted theta function $\theta_{E_6, P}$ with $\deg P > 0$ is a $\Gamma_1(3)$ cusp form with a *double* zero at $z = 0$, so vanishes as long as $3 + \deg P < 9$. For each of A_2, E_6, K_{12} the 5-design property can also be deduced from the action of the automorphism group, but (as we saw in levels 1 and 2) these

spherical-design properties also persist for lattice in higher dimension whose symmetry groups have more invariants of degree < 6.

Recall that for level-2 lattices we made use of linear combinations of Eisenstein series $\mathsf{E}_k(z)$ and $\mathsf{E}_k(2z)$ to give exact or approximate formulas for the representation numbers $N_{2n}(L)$. We can give some similar formulas here. For example, $M_2(\Gamma_1(3))$ is generated by

$$E_2(z) = \frac{1}{2}\big(3\mathsf{E}_2(3z) - \mathsf{E}_2(z)\big) = 1 + 12 \sum_{m=1}^{\infty} \left(\frac{mq^m}{1-q^m} - \frac{3mq^{3m}}{1-q^{3m}} \right), \tag{6}$$

which must equal $\theta_{A_2}^2$, whence $N_{2n}(A_2^2) = 12\sigma_1(n) - 36\sigma_1(n/3)$ for every positive integer n; and $M_4(\Gamma_1(3))$ is generated by $\mathsf{E}_4(z)$ and $\mathsf{E}_4(3z)$, so

$$\theta_{A_2^4}(z) = \frac{1}{10}\big(\mathsf{E}_4(z) + 9\mathsf{E}_4(3z)\big), \quad \theta_{A_2 E_6}(z) = \frac{1}{40}\big(13\mathsf{E}_4(z) + 27\mathsf{E}_4(3z)\big), \tag{7}$$

and from $\theta_{A_2 E_6}$ we also deduce the formula $\frac{1}{40}\big(\mathsf{E}_4(z) + 39\mathsf{E}_4(3z)\big)$ for the lattice $(A_2 E_6)' = A_2 \oplus E_6'$, and again recover formulas for $N_{2n}(A_2^4)$, $N_{2n}(A_2 E_6)$, $N_{2n}(A_2 \oplus E_6')$ as sums over divisors of n, e.g.

$$N_{2n}(A_2^4) = 24\big(\sigma_3(n) + 9\sigma_3(n/3)\big), \quad N_{2n}(A_2 E_6) = 6\big(13\sigma_3(n) + 27\sigma_3(n/3)\big). \tag{8}$$

The lattice $A_2 E_6$ is characterized by its theta function (and even by its rank 8 and root number of 78); but for A_2^4 we find another lattice $\big(D_4 \oplus (D_4\langle 3\rangle)\big)^{++}$ that has the same theta function, and thus yields our first example of isospectral tori in dimension as low as 8.

But in level 3 we also have lattices of rank $2k$ with k odd, and their theta functions cannot be obtained in the same way from $\Gamma(1)$ Eisenstein series. It is true that we can write formulas such as

$$\theta_{A_2}(z) = E_2(z)^{1/2} = 1 + 6q + 6q^3 + 6q^4 + 12q^7 + 6q^9 + 6q^{12} + 12q^{13} + 6q^{16} + \cdots, \tag{9}$$

but this does not give us access to the representation numbers $N_{2n}(A_2)$. Still, there is a known formula for $N_{2n}(A_2)$ as a divisor sum, using the arithmetic of the cyclotomic number field $\mathbf{Q}(\sqrt{-3}) = \mathbf{Q}(e^{2\pi i/3})$ (because the norms of A_2 are the integers $2(x^2 - xy + y^2)$, and $x^2 - xy + y^2$ is also the algebraic norm of a general algebraic integer $x + ye^{2\pi i/3}$ in that cyclotomic field):

$$N_{2n}(A_2) = 6 \sum_{d|n} \chi_3(d), \tag{10}$$

where χ_3 is again the nontrivial Dirichlet character mod 3. The coefficients of θ_{E_6} suggest similar patterns; for example, when p is prime, $N_{2p}(E_6) = 72(p^2 + 1)$ for $p \equiv 1 \bmod 3$ and $N_{2p}(E_6) = 90(p^2 - 1)$ for $p \equiv -1 \bmod 3$ (and also for $p = 3$). This suggests that the construction of Eisenstein series might be modified to produce modular forms of odd weight for $\Gamma_1(3)$ whose coefficients are even-power divisor sums weighted by χ_3, and more generally modular forms of level N whose coefficients are divisor sums weighted by the Nebentypus character. We next carry out such a construction.

Recall that for even k the Eisenstein series of weight k for $\Gamma(1)$ is proportional to $\sum_{m,n}(mz + n)^{-k}$, the sum ranging over all $(m, n) \neq (0, 0)$. We generalize as follows: fix a prime[1] $N > 1$, a nontrivial

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[1]The construction generalizes to non-prime levels N and characters of conductor N, but in that setting there are more cusps to check, especially when the level is not squarefree; for now we introduce the idea for prime N only.

Dirichlet character $\chi \bmod N$, and an integer $k > 2$ such that $(-1)^k = \chi(-1)$, define

$$G_k(z, \chi) = \sum_{\substack{m,n \in \mathbf{Z} \\ (m,n) \neq (0,0)}} \overline{\chi}(n)(Nmz + n)^{-k}. \tag{11}$$

The sum converges absolutely because $k > 2$, and the parity condition $(-1)^k = \chi(-1)$ is needed to prevent the sum from simplifying to zero due to cancellation between the (m, n) and $(-m, -n)$ terms. For $g = \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \in \Gamma_1(n)$ we have

$$G_k(gz, \chi) = (cz + d)^k \sum_{\substack{m,n \in \mathbf{Z} \\ (m,n) \neq (0,0)}} \overline{\chi}(n)\big((Nam + cn)z + (Nbm + dn)\big)^{-k}. \tag{12}$$

Since $N|c$, the coefficient $Nam + cn$ of z is a multiple of N; combining this with $\det(g) = 1$, we see that as (m, n) ranges over all nonzero integer vectors in $\mathbf{Z}^2$, so does

$$(m', n') := (Nam + cn, Nbm + dn).$$

But the constant coefficient $n' = Nbm + dn$ is congruent to $dn \bmod N$, so $\chi(n') = \chi(d)\chi(n)$. Therefore

$$G_k(gz, \chi) = \chi(d)\,(cz + d)^k \sum_{\substack{m',n' \in \mathbf{Z} \\ (m',n') \neq (0,0)}} \overline{\chi}(n')(mz + n)^{-k} = \chi(d)\,(cz + d)^k G_k(z, \chi). \tag{13}$$

Thus $G_k(z, \chi)$ is at least weakly modular of level N with Nebentypus χ. In particular, $G_k(z, \chi) = G_k(z + 1, \chi)$ so $G_k(z, \chi)$ has an expansion in powers of $q = e^{2\pi i z}$. The constant coefficient is

$$G_k(i\infty, \chi) = 2 \sum_{n=1}^{\infty} \overline{\chi}(n)n^{-k} = 2L(k, \overline{\chi}), \tag{14}$$

which is known to be an algebraic multiple of π^k under our hypothesis that $(-1)^k = \chi(-1)$; for example,

$$L(3, \chi_3) = 4\pi^3/3^{9/2}, \quad L(5, \chi_3) = 4\pi^5/3^{13/2}, \quad L(7, \chi_3) = 56\pi^7/3^{19/2}, \tag{15}$$

and in general $L(k, \overline{\chi})$ is a rational multiple of $N^{1/2}\pi^k$ if χ is a real character. By (14) we have

$$
\begin{aligned}
G_k(z, \chi) &= 2L(k, \overline{\chi}) + 2 \sum_{m=1}^{\infty} \sum_{n \in \mathbf{Z}} \overline{\chi}(n)(Nmz + n)^{-k} \\
&= 2L(k, \overline{\chi}) + 2 \sum_{m=1}^{\infty} \sum_{n_0 \bmod N} \overline{\chi}(n_0) \sum_{n \equiv n_0 \bmod n} (Nmz + n)^{-k}.
\end{aligned} \tag{16}
$$

The innermost sum is

$$N^{-k} \sum_{n_1 \in \mathbf{Z}} (mz + (n_0/N) + n_1)^{-k} = \frac{(-2\pi i)^k}{(k-1)!N^k} \sum_{d=1}^{\infty} d^{k-1} e^{2\pi i d n_0/N} q^{md} \tag{17}$$

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where $q = e^{2\pi i z}$ (for the last step, see formula (32) on page 92 of Serre's *A Course in Arithmetic*). Multiplying this by $\overline{\chi}(n_0)$ and summing over n_0 yields

$$\frac{(-2\pi i)^k}{(k-1)!N^k} \sum_{d=1}^{\infty} d^{k-1}\tau_d(\overline{\chi})q^{md}, \tag{18}$$

where $\tau_d(\overline{\chi})$ is the *Gauss sum*

$$\tau_d(\overline{\chi}) := \sum_{n_0 \bmod N} \overline{\chi}(n_0)e^{2\pi i d n_0/N} = \chi(d)\tau_1(\overline{\chi}). \tag{19}$$

It is known that $\tau_1(\overline{\chi})$ is a complex number of absolute value $N^{1/2}$; for example,[2]

$$\tau_1(\chi_3) = e^{2\pi i/3} - e^{-2\pi i/3} = \sqrt{3}\,i. \tag{20}$$

Putting everything together, we find that

$$G_k(z, \chi) = 2L(k, \overline{\chi}) + 2\frac{(-2\pi i)^k \tau_1(\overline{\chi})}{(k-1)!N^k} \sum_{m=1}^{\infty} \sum_{d=1}^{\infty} \chi(d)d^{k-1}q^{md}. \tag{21}$$

For example,

$$G_3(z, \chi_3) = \frac{8\pi^3}{3^{9/2}} \left(1 - 9 \sum_{m=1}^{\infty} \sum_{d=1}^{\infty} \chi_3(d)d^{k-1}q^{md} \right) = \frac{8\pi^3}{3^{9/2}}[1 - 9(q + 3q^2 - q^3 - 13q^4 \cdots)]. \tag{22}$$

The bracketed series agrees with the linear combination $(3\theta_{A_2}^3 - \theta_{E_6})/2$ of our generators of $M_3(\Gamma_1(3))$.

We check the growth condition at the cusp $z = 0$, and obtain an independent modular form of weight k, by applying w_N, which takes $G_k(z, \chi)$ to a multiple of

$$z^{-k}G_k(-1/(Nz), \chi) = \sum_{\substack{m,n \in \mathbf{Z} \\ (m,n) \neq (0,0)}} \overline{\chi}(n)(nz - m)^{-k}. \tag{23}$$

This is simpler: the constant term vanishes because $\overline{\chi}(n) = 0$ for $n = 0$, and the $n \neq 0$ terms are

$$2\sum_{n=1}^{\infty} \overline{\chi}(n) \sum_{m \in \mathbf{Z}} (nz - m)^k = 2\frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \overline{\chi}(n) \sum_{d=1}^{\infty} d^{k-1}q^{nd}. \tag{24}$$

Removing the factor $2(-2\pi i)^k/(k-1)!$ leaves the series whose q^n coefficient is $\sum_{d|n} \overline{\chi}(n/d)d^{k-1}$. For example, $w_3 G_3(z, \chi)$ is a multiple of

$$q + 3q^2 + 9q^3 + 13q^4 + 24q^5 + 27q^6 + 50q^7 + \cdots. \tag{25}$$

Multiplying (25) by 9, 27, or 81 and adding to the series $1 - 9(q + 3q^2 - q^3 - 13q^4 \cdots)$ of (22) recovers the theta functions of E_6', A_2^3, E_6 respectively. For example, $N_{2n}(E_6') = 9\sum_{d|n}(\chi_3(n/d) - \chi_3(d))$, which includes the formula for $N_{2n}(E_6)$ (as $N_{6n}(E_6')$) and vanishes as it should for $n \equiv 1 \bmod 3$.

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[2] In general if χ is a real character to prime modulus N then $\tau_1(N) = N^{1/2}$ or $N^{1/2}i$ according to whether $N \equiv +1$ or $-1 \bmod 4$.