

Math 272y: Rational Lattices and their Theta Functions

11 November 2019:

Statements of general modularity results for weighted theta functions;
lattices of level N and their theta functions

So far we have largely worked with the theta functions θ_L and $\theta_{L,P}$ of self-dual lattices $L \subset \mathbf{R}^n$, which we showed are modular forms of weight $\frac{n}{2} + \deg(P)$. Our main remaining goal for this course is to prove and apply the following generalization:

Theorem 1. *If $L \subset \mathbf{R}^n$ is any rational lattice, $f : L \rightarrow \mathbf{C}$ any periodic function, and $P : \mathbf{R}^n \rightarrow \mathbf{C}$ any harmonic polynomial of degree d , then*

$$\sum_{v \in L} f(v) P(v) \exp(\pi i \langle v, v \rangle z) \tag{1}$$

is a modular form of weight $\frac{n}{2} + \deg(P)$ for some congruence subgroup of $\mathrm{SL}_2(\mathbf{Z})$, and is a cusp form if P is nonconstant.

The statement of this theorem contains several terms which may be unfamiliar:

- A “rational lattice” is a lattice for which $\langle \cdot, \cdot \rangle$ takes values in \mathbf{Q} .
- A “periodic function” f on a lattice L is one for which there exists a finite-index sublattice L_0 such that $f(v) = f(v')$ for all $v, v' \in L$ with $v \equiv v' \pmod{L_0}$.
- A “congruence subgroup” is a subgroup (necessarily of finite index) containing $\Gamma(N)$ for some $N \geq 1$, where $\Gamma(N)$ is the normal subgroup

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a \equiv d \equiv 1, b \equiv c \equiv 0 \pmod{N} \right\} \tag{2}$$

of $\Gamma(1) = \mathrm{SL}_2(\mathbf{Z})$, which is the kernel of the reduction map $\mathrm{SL}_2(\mathbf{Z}) \rightarrow \mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})$.

We might not have defined “modular form for Γ ” for a general subgroup Γ of $\Gamma(1)$. These are holomorphic functions on \mathcal{H} that transform under Γ and satisfy a growth condition at $i\infty$ and at the images of $i\infty$ under $\Gamma(1)$. For an integer k , a holomorphic function $f : \mathcal{H} \rightarrow \mathbf{C}$ is a “weakly modular form of weight k ” for Γ if

$$f\left(\frac{az + b}{cz + d}\right) = (cz + d)^k f(z) \tag{3}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Such f is a *modular form of weight k* for Γ if moreover

$$(cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right) \text{ is bounded as } z \rightarrow i\infty \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1). \tag{4}$$

In that case f is a *cuspidal form of weight k* for Γ if furthermore

$$(cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right) \rightarrow 0 \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1). \quad (5)$$

Note that the transformation (3) is required only for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, but the growth conditions (4,5) are required for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$. Given (3), we need only check (4) or (5) for representatives of the orbits of Γ on $\{i\infty\} \cup \mathbf{Q}$, which are called the *cusps* of Γ , or of the “modular curve” \mathcal{H}/Γ ; there are finitely many cusps if $[\Gamma(1) : \Gamma] < \infty$. If Γ contains the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and k is odd then (3) implies that $f = 0$, but when $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \notin \Gamma$ there can be nonzero modular forms of odd weight.

Since the statement of Theorem 1 allows for odd n , we also need a definition of modular forms and cuspidal forms of weight k for half-integral k . The growth conditions (4,5) are the same (and are well-defined despite the sign ambiguity in $(cz + d)^k$). In the transformation formula (3) we must insert a factor $\varepsilon_{c,d} \in \mu_8$ (some 8th root of unity) whose exact dependence on c, d is somewhat subtle — a subtlety we encountered already in the transformation formulas for the $\theta_{\mathbf{Z}}$, which is a modular form of weight $1/2$ for the congruence group Γ_+ . Since we want θ_L to be modular for any lattice L , and the theta function of a direct sum $L_1 \oplus L_2$ is $\theta_{L_1}\theta_{L_2}$, we simply define f to be weakly modular of half-integral weight k for Γ if and only if $\Gamma \subseteq \Gamma_+$ and $\theta_{\mathbf{Z}}f$ is weakly modular of weight $k + \frac{1}{2}$ for Γ .

It will be a while before we prove the modularity of (1) in full generality. For now we give the first step in our proof of Theorem 1, and then introduce the important special case where f is constant and L is an even lattice.

We claim that it is enough to prove Theorem 1 when $L = L_1^*$ for some even lattice L_1 and $f : L \rightarrow \mathbf{C}$ is constant on cosets of L_1 in L . Indeed if $L \supseteq L_0$ and $f : L \rightarrow \mathbf{C}$ is L_0 -periodic, then f is also L_1 -periodic for any finite-index sublattice L_1 of L_0 ; and (1) can also be written as a sum over $v \in L'$ for any finite-index superlattice L' of L (by setting $f(v) = 0$ for $v \notin L$, which retains the L_0 -periodicity of f). So it is enough to prove that there is some finite-index even sublattice L_1 of L_0 such that $L_1^* \supseteq L$. But this is easy: we ask that $L_1 \subseteq L_0$ and $L_1 \subseteq L^*$; but L is rational, so L contains $L^* \cap L_0$ with finite index, and $L^* \cap L_0$ is integral because its dual contains L , and thus *a fortiori* contains L_0 . If $L^* \cap L_0$ happens to be even, we are done; else let L_1 be the even sublattice of $L^* \cap L_0$.

When we eventually prove Theorem 1, we shall fix L_1 and P , and consider the action of $\Gamma(1)$ on the $\text{disc}(L_1)$ -dimensional space consisting of functions (1) for *all* L_1 -periodic functions $L_1^* \rightarrow \mathbf{C}$. We introduce this technique by considering the special case of $\theta_{L,P}$ for any even lattice L .

If L is an even lattice then $\theta_{L,P}$ is invariant under $T : z \mapsto z + 1$. Combined with Theorem 1, this will make $\theta_{L,P}$ modular for the subgroup of $\Gamma(1)$ generated by $\Gamma(N)$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which is the congruence subgroup

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbf{Z}) : a \equiv d \equiv 1, c \equiv 0 \pmod{N} \right\} \quad (6)$$

of $\Gamma(1)$, often abbreviated “ $\left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N \right\}$ ”. This is a normal subgroup of $\Gamma_0(N)$, defined as

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) : c \equiv 0 \pmod N \right\} \quad (7)$$

(so “ $\left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N \right\}$ ”); the homomorphism $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto [d]$ descends to the quotient map

$$\Gamma_0(N)/\Gamma_1(N) \cong (\mathbf{Z}/N\mathbf{Z})^*.$$

Note that while $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \notin \Gamma_1(N)$ once $N > 2$, the group $\Gamma_0(N)$ does contain $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ for all N , so if we map $\Gamma_0(N)$ and $\Gamma_1(N)$ to subgroups of $\mathrm{PSL}_2(\mathbf{Z})$ then their quotient is $(\mathbf{Z}/N\mathbf{Z})^*/\{\pm 1\}$.

It may happen that for some modular form f for $\Gamma_1(N)$ there is a character $\chi : (\mathbf{Z}/N\mathbf{Z})^* \rightarrow \mathbf{C}^*$ such that f also transforms under $\Gamma_0(N)$ via

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z); \quad (8)$$

then f is said to be a modular form of weight k and “level N with Nebentypus χ ”. In particular, a modular form for $\Gamma_0(N)$ has level N with trivial Nebentypus.

If L is an even lattice then each $\theta_{L,P}$ is modular of level N for some N described as follows:

Theorem 2. *Suppose $L \subset \mathbf{R}^n$ is an even lattice, $N \geq 1$ an integer such that $L' := (L^*)\langle N \rangle$ is also an even lattice, and $P : \mathbf{R}^n \rightarrow \mathbf{C}$ is any harmonic polynomial of degree d . Then $\theta_{L,P}$ is a modular form of weight $\frac{n}{2} + \deg(P)$ for $\Gamma_0(N)$, and is a cusp form if P is nonconstant.*

(As with Theorem 1, it will be a while before we give a complete proof.) We shall thus say that if L is a nondegenerate even lattice then the *level* of L is the smallest integer $N \geq 1$ such that $(L^*)\langle N \rangle$ is also an even lattice; equivalently, N is the least common denominator of the rational numbers $\frac{1}{2}\langle v, v \rangle$ for $v \in L^*$. For example, L has level 1 if and only if L is self-dual; the root lattice A_n has level $n+1$ or $2(n+1)$ according as n is even or odd; D_n has level 8, 4, 2 according as n is odd, singly even, or doubly even; E_n has level 3, 4, 1 for $n = 6, 7, 8$; and if even lattices L_1 and L_2 have levels N_1 and N_2 then $L_1 \oplus L_2$ has level $\mathrm{lcm}(N_1, N_2)$. Every prime factor of the level of L is also a factor of $\mathrm{disc}(L)$; more precisely, if L has level N and the discriminant group L^*/L has exponent e then $e|N$ and $N|e^2$. To prove $e|N$: since $(N/2)\langle v, v \rangle \in \mathbf{Z}$ for all $v \in L^*$ it follows that $N\langle v, w \rangle \in \mathbf{Z}$ for all $v, w \in L^*$; but then for any $w \in L$ we have $\langle v, Nw \rangle \in \mathbf{Z}$ for all $v \in L^*$, so $Nw \in (L^*)^* = L$, whence $NL^* \subseteq L$ and $N(L^*/L) = \{0\}$. For $N|e^2$: for any $v \in L^*$ we have $ev \in L$, so $e^2\langle v, v \rangle = \langle ev, ev \rangle \in 2\mathbf{Z}$, so e^2L^* is an even lattice and we are done.

Crucially, if L has level N then so does $L' := (L^*)\langle N \rangle$. To see this, first note that

$$L' := (L^*)\langle N \rangle \iff L := (L'^*)\langle N \rangle \quad (9)$$

(this holds for any N and is not special to the level). Thus N is an integer such that $(L'^*)\langle N \rangle$ is an even lattice, so the level of L' is a factor of N ; but since L and L' occur symmetrically in (9) the level of L' is also a multiple of N , so the two levels are equal.

For $N = 1$, Theorem 2 states what we already know about $\theta_{L,P}$ for even self-dual lattices L . With a little more work we can adapt the proof to the cases $N = 2, 3, 4$. For $N = 1$, we saw that $\Gamma(1)$ is generated by $T : z \mapsto z + 1$ and $S : z \mapsto -1/z$, and showed that $\theta_{L,P}$ transforms under T because L is even, and under S by Poisson summation. For general N , it is still true that

$$\theta_{L,P}(z + 1) = \theta_{L,P}(z), \quad \theta_{L',P}(z + 1) = \theta_{L',P}(z); \quad (10)$$

but instead of $z \mapsto -1/z$ we are led to the involution $w_N : z \mapsto -1/(Nz)$, which switches the theta series for L and L' : taking¹ $d = \deg(P)$, we have

$$\begin{aligned} \theta_{L,P}(it) &= i^d \operatorname{disc}(L)^{-1/2} t^{-(n/2)-d} \theta_{L^*,P}(i/t) \\ &= i^d \operatorname{disc}(L)^{-1/2} t^{-(n/2)-d} N^{-d/2} \theta_{L',P}(i/Nt), \end{aligned} \quad (11)$$

and likewise

$$\theta_{L',P}(it) = i^d \operatorname{disc}(L)^{1/2} t^{-(n/2)-d} N^{-d/2} \theta_{L,P}(i/Nt); \quad (12)$$

and w_N switches it with $i/(Nt)$. (The factors $N^{-d/2}$ compensate for scaling L^* to $L' = (L^*)\langle N \rangle$ and $(L')^*$ to L .) Now the involution w_N , though not in $\Gamma_1(N)$ or even in $\Gamma(1)$ once $N > 1$, is in the normalizers of $\Gamma_1(N)$ and $\Gamma_0(N)$ in $\operatorname{SL}_2(\mathbf{R})$: we compute that if $g(z) = (az + b)/(cz + d)$ then

$$w_N(g(w_N(z))) = \frac{dz - (c/N)}{-Nbz + a},$$

whence if $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in $\Gamma_1(N)$ or $\Gamma_0(N)$ then $\begin{pmatrix} d & -c/N \\ -Nb & a \end{pmatrix}$ is in the same congruence group. Thus if we apply (11, 12), use the invariance of $\theta_{L,P}$ and $\theta_{L',P}$ under \mathbf{Z} -translations $T^b : z \mapsto z + b$, and apply (11, 12) again, we find that $\theta_{L,P}$ and $\theta_{L',P}$ both transform under the $\Gamma_1(N)$ element $z \mapsto z/(1 - Nbz)$: taking $z = it$, we find that

$$\theta_{L,P}(z) = i^{2d} (\operatorname{disc}(L) \operatorname{disc}(L'))^{-1/2} \left(\frac{z}{i} \frac{-1/Nz + b}{i} \right)^{-(n/2)-d} N^{-d} \theta_{L,P}(z/(1 - Nbz)). \quad (13)$$

Now the factor $i^{2d} = (-1)^d$ can be assumed to be $+1$, because if d is odd then $\theta_{L,P} = 0$ by symmetry. The factor $(\operatorname{disc}(L) \operatorname{disc}(L'))^{-1/2}$ is $N^{-n/2}$, and the product of z/i and $(-1/(Nz) + b)/i$ is $(1 - Nbz)/N$. Thus (13) simplifies to

$$\theta_{L,P}(z) = (1 - Nbz)^{-k} \theta_{L,P}(z/(1 - Nbz)). \quad (14)$$

Hence (3) holds for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -Nb & 1 \end{pmatrix}$, and therefore by induction also for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the subgroup of $\Gamma_1(N)$ generated by matrices $\begin{pmatrix} 1 & 0 \\ -Nb & 1 \end{pmatrix}$.

Unfortunately these transformations do not suffice to generate $\Gamma_1(N)$ once $N > 4$. Fortunately they do generate $\Gamma_1(N)$ for $N \leq 4$, which includes many interesting new examples starting with A_2 (of

¹Anselm Blumer notes that this notation unfortunately collides with an entry of our generic $\operatorname{SL}_2(\mathbf{Z})$ matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. There isn't quite a direct collision in any of the ensuing formulas, but I should change this eventually . . .

level 3) and D_4 (level 2). Moreover, in each of these cases $\Gamma_0(N) = \pm\Gamma_1(N)$, so any modular form for $\Gamma_1(N)$ is automatically a modular form for $\Gamma_0(N)$, possibly with nontrivial Nebentypus. Thus we obtain a proof of Theorem 2 in these three important special cases that is considerably simpler than the proofs of Theorems 1 and 2 in general. We do this by showing that $\Gamma_1(N)$ is a hyperbolic triangle group if (and only if) $N \leq 4$, and that the same is true of the group generated by $\Gamma_1(N)$ and w_N . This will also let us adapt some of our constructions and results for modular forms for $\Gamma(1)$ to the setting of lattices of levels 2, 3, and 4.

We have seen that $\mathcal{F} = \{z \in \mathcal{H} : |\operatorname{Re}(z)| \leq 1/2, |z| \geq 1\}$ is a fundamental domain for the group generated by S (which we could now also call “ w_1 ”) and T . Likewise the group generated by w_N and T has fundamental domain

$$\mathcal{F}_N := \{z \in \mathcal{H} : |\operatorname{Re}(z)| \leq 1/2, |z| \geq 1/\sqrt{N}\}. \quad (15)$$

Once $N > 4$, the hyperbolic area of \mathcal{F}_N is infinite, so it cannot be a fundamental domain for $\langle \Gamma_1(N), w_N \rangle$. But for $N = 2, 3, 4$ we see that \mathcal{F}_N is a hyperbolic triangle (with three vertices at infinity for $N = 4$, and one for $N < 4$), whence \mathcal{F}_N has finite area. It follows that $\mathcal{F}_N \cup w_N \mathcal{F}_N$ also has finite area; this is a fundamental domain for the index-2 subgroup $\langle w_N, T \rangle \cap \Gamma_1(N)$ of $\langle w_N, T \rangle$. In each case we find that the hyperbolic area of \mathcal{F}_N (which is half the hyperbolic area of $\mathcal{F}_N \cup w_N \mathcal{F}_N$) is $\pi/6$ times the index of $\Gamma_1(N)$ in $\Gamma(1)$, and thus that $\langle \Gamma_1(N), w_N \rangle$ is generated by w_N and T .

In fact in each of these cases both $\langle \Gamma_1(N), w_N \rangle$ and $\Gamma_1(N)$ are hyperbolic triangle groups. The “triangle” for $\langle \Gamma_1(N), w_N \rangle$ has an ideal vertex at $i\infty$, a right angle at i/\sqrt{N} , and a vertex at the intersection of the line $\operatorname{Re}(z) = -1/2$ with the semicircle $|z| = 1/\sqrt{N}$. This third vertex has imaginary part $1/2, 1/\sqrt{12}, 0$ for $N = 2, 3, 4$ respectively, with angles $\pi/4, \pi/6$, and 0 . For $\Gamma_1(N)$, replace the second vertex i/\sqrt{N} by the reflection of the third vertex about the imaginary axis. These hyperbolic triangles have areas $\pi/2, 2\pi/3$, and π respectively, which is $3\pi/6, 4\pi/6, 6\pi/6$. For $\Gamma(1)$, the triangle has area $\pi/6$; and indeed $[\Gamma(1) : \Gamma_1(N)]$ is $3, 4, 6$ for $N = 2, 3, 4$. Hence each of these $\Gamma_1(N)$ is generated by T and the respective w_N . We have thus checked that $\theta_{L,P}$ is weakly modular of weight $(n/2) + d$ and level N when $N \leq 4$. For $N = 2$ and $N = 3$, the growth condition (4) or (5) need only be checked for $f(z)$ and $f(-1/z)$, which is immediate from the functional equations (11,12) relating $\theta_{L,P}$ with $\theta_{L',P}$. This completes the proof of Theorem 2 for $N = 2$ and $N = 3$. For $N = 4$ there is one more cusp to check, and we shall see that this verification is much the same as we did for $\theta_{L,P}$ when L is self-dual but possibly odd. For now, our next tasks are to describe the modular forms for $\Gamma_1(2)$ and $\Gamma_1(3)$, and to combine these with Theorem 2 to analyze lattices of levels 2 and 3.