

Math 55a: Honors Advanced Calculus and Linear Algebra

Metric topology VI: Cauchy sequences, completeness,
and a third formulation of compactness

Cauchy sequences and total boundedness. [See Simmons, pages 71–72 and 123 respectively.] We noted already that convergence of sequences is not an intrinsic notion: $(2^{-n}) = (1/2, 1/4, 1/8, \dots)$ converges as a sequence in $\mathbf{R}$ but not in $(0, 1)$. A closely related notion which is intrinsic is that of a *Cauchy sequence*. A sequence $\{p_n\}$ in a metric space X is Cauchy if for every $\epsilon > 0$ there is an integer N such that $d(p_m, p_n) < \epsilon$ for all $m, n \geq N$. (This can be rephrased as “ $d(p_m, p_n) \rightarrow 0$ as $m, n \rightarrow \infty$ independently.”) This notion is intrinsic because it does not require a limit point p whose existence may depend on the choice of ambient space. However, *any convergent sequence is Cauchy* [Simmons, p.71]: if $p_n \rightarrow p$, and $d(p_n, p) < \frac{1}{2}\epsilon$ for all $n \geq N$, then $d(p_m, p_n) < \epsilon$ for all $m, n \geq N$ by the triangle inequality.

Recall that X is said to be *totally bounded* if it has an ϵ -net for each $\epsilon > 0$ [Simmons, page 123]. We have:

Theorem. *X is totally bounded if and only if every sequence in X has a Cauchy subsequence.*

Proof: $(\Leftarrow)$ [This argument should look familiar.] We may assume $X \neq \emptyset$. Given $\epsilon > 0$, we inductively construct an ϵ -net $\{p_1, \dots, p_N\}$ as follows. Choose p_1 arbitrarily. Having chosen $p_1, \dots, p_n$, if $\{p_1, \dots, p_n\}$ is not yet an ϵ -net, let p_{n+1} be a point such that $d(p_m, p_{n+1}) \geq \epsilon$ for each $m = 1, \dots, n$. We claim that this process must terminate. Indeed, if it didn't, we would obtain a sequence $\{p_n\}_{n=1}^\infty$ any two of whose points are at distance $\geq \epsilon$ from each other; but such a sequence clearly has no Cauchy subsequence.

$(\Rightarrow)$ Let S_k be a $(1/2k)$ -net for $k = 1, 2, 3, \dots$. Given a sequence $\{p_n\}$ in X , extract a subsequence $\{q_n^{(1)}\}$ contained in a radius- $(1/2)$ neighborhood about one of the points in S_1 . From $\{q_n^{(1)}\}$ extract a subsequence $\{q_n^{(2)}\}$ contained in a radius- $(1/4)$ neighborhood about one of the points in S_2 . Keep going inductively: $\{q_n^{(k)}\}$ is a subsequence of $\{q_n^{(k-1)}\}$ contained in a radius- $(1/2k)$ neighborhood about one of the points in S_k . This is possible because for each k the sequence $\{q_n^{(k-1)}\}$ is infinite while S_k is finite. Note that $d(q_n^{(k)}, q_{n'}^{(k)}) < 1/k$ for all k, n, n' . Now consider the *diagonal subsequence* $\{q_k^{(k)}\}$. This is a subsequence of the original $\{p_n\}$; moreover, if $m, n \geq N$ then $q_m^{(m)}$ and $q_n^{(n)}$ are both contained in the subsequence $\{q_n^{(N)}\}$, and thus are at distance $< 1/N$. Thus $\{q_k^{(k)}\}$ is a Cauchy subsequence of $\{p_n\}$. $\square$

Completeness and Compactness. A metric space X is said to be *complete* if every Cauchy sequence in X converges in X . If X, Y are nonempty metric spaces then $X \times Y$ is complete if and only if X and Y are complete. Similarly,

if S is a nonempty set then $\mathcal{B}(S, Y)$ is complete if and only if Y is. If X is complete, then a subset $E \subseteq X$ is complete if and only if it is closed. Indeed we have seen that E is closed iff, for every sequence $\{p_n\}$ in E converging to some $p \in X$, its limit p is also contained in E . But by hypothesis $\{p_n\}$ converges in X if and only if it is Cauchy — and since that's an intrinsic notion, $\{p_n\}$ is Cauchy in X if and only if it is Cauchy in E . So, E is closed iff every Cauchy sequence in E converges in E , which is precisely the condition for E to be complete. As a corollary, if Y is complete and X is any topological space, $\mathcal{C}(X, Y)$ is complete, because we have already seen that it is closed as a subset of $\mathcal{B}(X, Y)$.

It is a fundamental fact that $\mathbf{R}$ is complete — from which it follows that $\mathbf{R}^k$ ($k = 1, 2, 3, \dots$), and any closed subset of $\mathbf{R}^k$, is complete. Moreover, any bounded subset of $\mathbf{R}^k$ is totally bounded, because such a set is contained in some $B_M(x)$, and the set of $y \in B_M(x)$ such that each coordinate of $y - x$ is in $\epsilon\mathbf{Z}$ is an ϵ -net for each $\epsilon > 0$. [Note that we're implicitly using the result of Problem 7 in the third homework set here: if $E \subseteq X$ is totally bounded relative to X , then E is totally bounded.] This, together with our above characterization of totally bounded sets, yields the **Heine-Borel Theorem**:¹ A subset of $\mathbf{R}^k$ is compact if and only if it is closed and bounded. More generally, we have our third equivalent definition of compactness, in the context of subsets of a complete metric space:

Theorem. A subset of a complete metric space is compact if and only if it is closed and totally bounded.

Proof: Given our work thus far we need only show that a sequentially compact metric space is complete. But if $\{p_n\}$ is a Cauchy sequence then the limit of any convergent subsequence is also the limit of the entire sequence. $\square$

Applications to continuous functions. We noted already that the continuous image of a compact set is compact. In particular it is bounded and closed. So, for instance, if X is compact and Y is any metric space then $\mathcal{C}(X, Y)$ consists simply of all continuous functions $f : X \rightarrow Y$, because any such function is automatically bounded. In the important special case $Y = \mathbf{R}$, we have seen that bounded closed sets contain their sup and inf; so, for instance, any continuous real-valued function on a nonempty compact space attains its supremum and infimum. We shall use this repeatedly to prove many results from Rolle's theorem (which undergirds the entire theory of Taylor expansions) to the spectral theorem for Hermitian operators on finite-dimensional inner product spaces. Going in another direction, for any $f, g \in \mathcal{C}(X, Y)$ we may consider the continuous function $x \mapsto d(f(x), g(x))$ to show that when X is compact the distance in $\mathcal{C}(X, Y)$ is given by $d(f, g) = \max_{x \in X} d(f(x), g(x))$ [NB the usual sup, to substitute for max when X is infinite, is not needed in the compact case].

If X, Y are metric spaces with X compact then any continuous $f : X \rightarrow Y$ is

¹As Simmons suggests on page 114, the proof given there is more complicated than we'll need; the usual approach is to do what we've done here via ϵ -nets or something equivalent.

uniformly continuous. Indeed, given $\epsilon > 0$ the preimages of $(\epsilon/2)$ -neighborhoods in Y constitute an open cover of X , which has a Lebesgue number $\delta > 0$ by the Lebesgue covering lemma.² Thus each $f(B_\delta(x))$ ($x \in X$) is contained in some $B_{\epsilon/2}(y)$, so $d(x, x') < \delta \Rightarrow d(f(x), f(x')) < \epsilon$ as claimed.

Completions. How did we know in the first place that $\mathbf{R}$ is complete? The answer to that depends on how you define the real numbers — various approaches make this result a theorem, an axiom, or even a definition. If you believe in $\sup$ and $\inf$, you can obtain the limit of a real Cauchy sequence $\{p_n\}$ as either $\limsup p_n$ or $\liminf p_n$, these being defined for any bounded sequence of $s_n \in \mathbf{R}$ by

$$\limsup p_n := \inf_{n=1}^{\infty} \left(\sup_{m \geq n} s_m \right), \quad \liminf p_n := \sup_{n=1}^{\infty} \left(\inf_{m \geq n} s_m \right).$$

(Note that the sequences $\{\sup_{m \geq n} s_m\}_{n=1}^{\infty}$ and $\{\inf_{m \geq n} s_m\}_{n=1}^{\infty}$ are monotonically decreasing and increasing respectively).

One nice approach is to obtain $\mathbf{R}$ as a “completion” of $\mathbf{Q}$. This approach is implicit in writing an arbitrary real number as a possibly nonterminating decimal, with an ambiguity in cases such as $54.9999\dots = 55.0000\dots$. Once this is done, it can be generalized by the following important construction.³ To any metric space X we associate its *completion*, which is a complete metric space X^* in which X is embedded isometrically as a dense subset. We construct X^* out of Cauchy sequences in X . For any two such sequences $\{p_n\}, \{q_n\}$, the sequence $\{d_n\} := \{d(p_n, q_n)\}$ in $\mathbf{R}$ is Cauchy, and thus has a limit, which we call the “distance” between $\{p_n\}$ and $\{q_n\}$. This “distance” satisfies all the axioms of a distance function, except that $d(\{p_n\}, \{q_n\}) = 0$ need not imply $\{p_n\} = \{q_n\}$. But we get a genuine metric space by identifying any Cauchy sequences at distance 0 from each other. The resulting space is the completion X^* ; it contains an isometric copy of X consisting of the equivalence classes of constant sequences $x, x, x, \dots$. It is easy to see that this copy of X is dense in X^* . The only tricky thing to prove is that X^* is in fact complete: we must show that X^* contains limits of Cauchy sequences not only in X , but also in X^* . Suppose $\{x_n^*\}$ is such a Cauchy sequence. We find a Cauchy sequence in X whose limit in X^* is also the limit of $\{x_n^*\}$. For $k = 1, 2, 3, \dots$ choose N_k so that $d(x_m^*, x_n^*) < 1/k$ for all $m, n \geq N_k$. Let $x_k \in X$ be chosen so $d(x_k, x_{N_k}^*) < 1/k$. One readily checks that $\{x_k\}$ is a Cauchy sequence at distance zero from $\{x_n^*\}$.

²See Simmons, page 122. We use the following form (do you see why it is equivalent with Simmons’ statement?): Let $\{U_\alpha\}$ be an open cover of the compact metric space X . Then there exists $r > 0$ (a “Lebesgue number” for the covering) such that, for every $x \in X$, the r -ball $B_r(x)$ is contained in some U_α . This is proved by contradiction. Assume no such r exists. Find a sequence $\{x_n\}$ in X such that $B_{1/n}(x_n)$ is contained in no U_α . Let x be the limit of a convergent subsequence. Then x is contained in some U_α ; but U_α is open, so x contains some $B_\rho(x)$ with $\rho > 0$. Now check that $B_\rho(x)$ contains some $B_{1/n}(x_n)$ to get the desired contradiction.

³Simmons’ construction in Problem 5 on pages 84–85, finding the completion of X in $\mathcal{B}(X, \mathbf{R})$, is somewhat idiosyncratic. Simmons does provide Problem 3 on page 58 which describes the construction of a metric space from a “pseudo-metric space” which satisfies all our axioms except for $d(p, q) = 0 \Rightarrow p = q$.

[This proof may also be viewed as a disguised “diagonal argument”. Say that $\{x_i\}$ is a “fast Cauchy sequence” if $d(x_m, x_n) < 1/N$ whenever $m, n > N$. Clearly every Cauchy sequence has a fast subsequence. In particular, each $x^* \in X^*$ is represented by a fast Cauchy sequence in X . Now suppose $\{x_n^*\}$ is a Cauchy sequence in X^* . For each n , choose a representative fast Cauchy sequence $\{x_{n,i}\}_{i=1}^\infty$. We claim: the diagonal sequence $\{x_{n,n}\}_{n=1}^\infty$ is a (not necessarily fast) Cauchy sequence in X whose limit is also the limit of $\{x_n^*\}$. To show that it’s Cauchy we argue in much the same way that we proved the continuity of a uniform limit of continuous functions. For large enough N , we have

$$d(x_{m,m}, x_{n,n}) \leq \frac{1}{m} + \frac{1}{n} + d(x_m^*, x_n^*) < \frac{2}{N} + d(x_m^*, x_n^*)$$

for all $m, n > N$. Now choose N large enough that $1/N < \epsilon/3$ and that $d(x_m^*, x_n^*) < \epsilon/3$ for all $m, n > N$. Then $d(x_{m,m}, x_{n,n}) < \epsilon$. Now the distance between the Cauchy sequences $\{x_{n,n}\}$ and $\{x_n^*\}$ vanishes⁴ because $d(x_{n,n}, x_n^*) \leq 1/n$. Hence they have the same limit. We have thus exhibited a limit of an arbitrary Cauchy sequence in X^* , completing the proof that X^* is complete.]

For instance, $\mathbf{R}$ is the completion of $\mathbf{Q}$, and $[0, 1]$ is the completion of $(0, 1)$. More generally, if X is already complete and E is any subset of X then the completion of E is naturally identified with the closure of E in X .

Example: N -adic numbers. There are other nice spaces that can be obtained by completing $\mathbf{Q}$ relative to a more exotic metric. For instance, suppose we say that two rational numbers r, r' are close to each other if their decimal expansions agree to many places to the *left* of the decimal point; more precisely, we define the “10-adic metric” d_{10} on $\mathbf{Q}$ as follows: if $r \neq r'$ we declare $d_{10}(r, r') = 10^{-e}$ where $r - r' = 10^e m/n$ for some integers m, n with n relatively prime to m as well as to 10. Of course $d_{10}(r, r) = 0$ for all r . One easily verifies that this is in fact a metric on $\mathbf{Q}$, and indeed satisfies not only the triangle inequality but the stronger “nonarchimedean triangle inequality” $d(p, q) \leq \max(d(p, r), d(q, r))$. For instance, with this metric Euler’s nonsensical series $1! - 2! + 3! - 4! + \dots$ actually converges (while the usual series $\sum_{n=1}^\infty 1/n!$ for e badly diverges). Completing $\mathbf{Q}$ relative to d_{10} yields the “10-adic numbers” $\mathbf{Q}_{10}$. Of course there is nothing special about 10 here: one can likewise define N -adic numbers for any fixed integer $N > 1$. In fact the nicest case is d_N for N prime: one can add, subtract, and divide in $\mathbf{Q}_N$ for all N , just as one does in $\mathbf{R}$, but one can only divide by arbitrary nonzero numbers if N is prime — do you see why? The N -adic numbers may appear to be a pathological curiosity (e.g. for each N they have the topology of a countable union of disjoint Cantor sets), and indeed they were originally constructed as an amusement; but N -adic numbers and generalizations thereof are now ubiquitous tools in number theory and algebraic geometry.

⁴In mathematical writing “ x vanishes” is a synonym for “ $x = 0$ ”.