

Math 55a: Honors Advanced Calculus and Linear Algebra

Revised for Math 25b, Spring [2013–]2014¹

Metric topology II: open and closed sets, etc.

Open balls (a.k.a. r -neighborhoods, open spheres) and open sets. To further study and make use of metric spaces we need several important classes of subsets of such spaces. They can all be based on the notion of the *open ball*, defined as follows. Let X be a metric space, $p \in X$, and $r > 0$. The *open ball* $B_r(p)$ with center p and radius r is the set of all $q \in X$ at distance $< r$ from p :

$$B_r(p) := \{q \in X : d(p, q) < r\}.$$

[Simmons (p.59) calls this the “open sphere” $S_r(p)$, and other texts from the same era (notably Rudin) call it the “ r -neighborhood” of p . Nowadays “sphere” usually means the set of points whose distance from p is *equal* to r , not less than r .] This term is motivated by the shape of $B_r(p)$ when X is \mathbf{R}^3 with the Euclidean metric. Here are some examples of $B_r(p)$ in other metric spaces: in \mathbf{R} , it is the “open interval” $(p - r, p + r)$; likewise in \mathbf{R}^n with the sup metric, $B_r(p)$ is an open (hyper)cube of side $2r$ centered at p ; if $d(\cdot, \cdot)$ is the discrete metric,² $B_r(p) = \{p\}$ or X according as $r \leq 1$ or $r > 1$. Visualizing $B_r(p)$ for various $p \in X$ and $r > 0$ is a good way to get a feel for the metric space X .

Now let G be any subset of X . The *interior points* of G are those $p \in X$ some neighborhood of which is contained in G , i.e. those $p \in X$ for which there exists $r > 0$ such that $B_r(p) \subseteq G$ [Simmons, pages 63–64]. Necessarily $p \in G$ (why?). The subset $G \subseteq X$ is said to be *open* in X if and only if every point of G is an interior point of G . Check that this is equivalent to the definition in Simmons, page 60; and that \emptyset and X itself are open in X (Simmons, Theorem A on page 60). Note that, unlike the notion of boundedness, openness of G depends not only on G but also on the “ambient space” X . For instance, every metric space is open as a subset of itself, but a one-point subset of \mathbf{R} cannot be open as a subset of \mathbf{R} (check these assertions!). We shall only say/write statements like “ G is open” when the ambient space is clear from context.

Calling $B_r(p)$ an “open ball” would be horribly confusing if such sets $B_r(p)$ could fail to be open. The name is justified by the following result (Simmons, Theorem B on p.61):

Theorem. *Every open ball is an open set.*

That is, for any metric space X , any $p \in X$, and any $r > 0$, the set $B_r(p)$ is open as a subset of X .

Proof: We must show that for any $q \in B_r(p)$ there is an $h > 0$ such that $B_h(q) \subseteq B_r(p)$. We claim that $h = r - d(p, q)$ works. Indeed, h is positive by the definition of $B_r(p)$; and for any $s \in B_h(q)$ we have $s \in B_r(p)$ because

$$d(p, s) \leq d(p, q) + d(q, s) < (r - h) + h = r,$$

¹but retaining some supplementary material that will not be covered in class.

²That is, $d(p, q) = 1$ for all distinct p, q (and necessarily $d(p, q) = 0$ when $p = q$).

so $B_h(q)$ is a subset of $B_r(p)$ as desired. \square

A key fact about open sets is that a finite intersection of open sets is again open, as is an *arbitrary* union of open sets (Simmons, Theorem D on pages 61–62):

Theorem. *i) if G_α is an open subset of X for each $\alpha \in I$, then so is their union $\cup_{\alpha \in I} G_\alpha$.*

ii) If each of G_1, \dots, G_n is an open subset of X , then so is their intersection $\cap_{i=1}^n G_i$.

Note that in part (i), I is an “index set” of arbitrary size, while in part (ii), it is essential that the intersection be finite — a counterexample with a countably infinite intersection is $X = \mathbf{R}$, $G_n = B_{1/n}(0) = (-1/n, 1/n)$ ($n = 1, 2, 3, \dots$), when $\cap_{i=1}^\infty G_n = \{0\}$ is not open.

Proof: (i) Put $G = \cup_{\alpha \in I} G_\alpha$. To show G is open, we must construct for each $x \in G$ a positive r such that $B_r(x) \subseteq G$. Since $x \in G_\alpha$ for some $\alpha \in I$, we already have $r > 0$ such that $B_r(x) \subseteq G_\alpha$. Since $G \supseteq G_\alpha$, it follows that $B_r(x) \subseteq G_\alpha$ as was needed.

(ii) Put $H = \cap_{i=1}^n G_i$. To show H is open, we must construct for each $x \in H$ a positive r such that $B_r(x) \subseteq H$, i.e. such that $B_r(x) \subseteq G_i$ for each $i = 1, \dots, n$. But each G_i is open, so we have r_1, \dots, r_n such that $B_{r_i}(x) \subseteq G_i$ for each i . Let $r = \min(r_1, \dots, r_n)$. Then $r > 0$ and $r \leq r_i$ for each i . Thus $B_r(x) \subseteq B_{r_i}(x)$, so $B_r(x) \subseteq G_i$, and we are done. \square

[For many purposes all that we’ll need to know about the family \mathcal{T} of open sets in X is that \mathcal{T} contains \emptyset and X , the intersection of any $G_1, \dots, G_n \in \mathcal{T}$, and an arbitrary union of $G_\alpha \in \mathcal{T}$. A family of subsets of a set X which satisfies these three conditions, whether or not it arises as the open sets of some metric space, is called a *topology* on X , which then becomes a *topological space* (X, \mathcal{T}) . See Chapter 3 of Simmons. Any result involving metric spaces which can be rephrased in terms of open sets and proved using only the above axioms on \mathcal{T} is then valid in the larger category of topological spaces.]

Closed sets and limit points. A *closed* subset of a metric space X is by definition³ the complement of an open subset. Using de Morgan’s laws (the complement of an intersection is the union of the complements, and vice versa; see Simmons, pages 11–12 [where A' means “the complement of A ”, a.k.a. A^c]), we immediately obtain:

Theorem. [Simmons, Theorem D on page 66]

i) if F_α is a closed subset of X for each $\alpha \in I$, then so is their intersection $\cap_{\alpha \in I} F_\alpha$.

ii) If each of F_1, \dots, F_n is a closed subset of X , then so is their union $\cup_{i=1}^n F_i$.

Unwinding the definition of a closed set, we see that $E \subseteq X$ is closed if and only if for every $p \notin E$ there exists $r > 0$ such that $B_r(p)$ is disjoint from E . (This is tantamount to Simmons’ definition on page 65.) The prototypical example of a closed set in X is the *closed ball* of radius $r \geq 0$ about a point $p \in X$, defined

³Simmons’ definition on page 65 is different; we reconcile them next page.

by

$$\bar{B}_r(p) := \{q \in X : d(p, q) \leq r\}$$

(As with the openness of $B_r(p)$, this requires proof; try to supply the proof yourself before consulting Theorem C in Simmons page 66.) Note that $r = 0$ is allowed, with $\bar{B}_0(p)$ being simply $\{p\}$. In \mathbf{R} , the closed r -ball about p is the “closed interval” $[p - r, p + r]$. Further examples of closed sets are \emptyset and X itself (Theorem A on page 65), and the complement $(B_r(p))^c = \{q \in X : d(p, q) \geq r\}$ of an open ball.

NB “closed” does not mean “not open”! A subset of a metric space might be both open and closed (as we already saw for \emptyset and X ; can you find more examples in a discrete set?). The subset can also fail to be either open or closed (as with a “half-open interval” $[a, b) \subset \mathbf{R}$, or more dramatically $\mathbf{Q} \subset \mathbf{R}$).

Simmons’ definition of a closed set involves the notion of a *limit point*. A point $p \in X$ is said to be a limit point of the subset $E \subseteq X$ if every open ball with center p contains a point of E other than p itself; i.e. if for all $r > 0$ there exists $q \in E$ such that $0 < d(p, q) < r$. Here is Simmons’ definition of a closed set; for us it is a theorem,⁴ since we must prove it is equivalent to our definition:

F is closed if and only if every limit point of F is contained in F.

Proof: Suppose F is closed, and let x be a limit point. We prove that $x \in F$ by contradiction. Assume that $x \notin F$. Since x would then be in the complement of F , it would be the center of an open ball $B_r(x)$ disjoint from F , contradicting the definition of a limit point. Therefore $x \in F$. We have thus shown that a closed set contains all its limit points.

Conversely, suppose F contains all its limit points. Then any $x \notin F$ is not a limit point of F . Thus there exists $r > 0$ such that $B_r(x)$ contains no point of F . Therefore F is closed. \square

An equivalent description of limit points is the following result:

Theorem. *p is a limit point of F if and only if there exist points $q_n \in F$ ($n = 1, 2, 3, \dots$), with each $q_n \neq p$, such that for every $r > 0$ we have $d(p, q_n) < r$ for all but finitely many n .*

Proof: (\Leftarrow) is clear, since “all but finitely many” certainly forces “at least one”. For (\Rightarrow) we construct q_n as follows: let $r = 1/n$ in the definition of limit point, and let q_n be a point such that $0 < d(p, q_n) < 1/n$. Then for each $r > 0$ we have $r > 1/N$ for some integer N ; then $d(p, q_n) < r$ once $n > N$, and there are only finitely many integers n which do not exceed N . \square

[We shall see that the q_n then constitute a *sequence* of points in $F \setminus \{p\}$ whose *limit* is p , once we define “sequence” and “limit” a few lectures hence.]

We also find:

⁴Simmons’ Theorem B on page 65, that a set is closed (in his sense) if and only if its complement is open, is logically equivalent.

Theorem. *A finite set has no limit points.*

Indeed, if F is finite then for each $p \in X$ there are only finitely many $q \neq p$ in F , and thus finitely many distances $d(p, q)$. Thus if r is smaller than the least of them then there is no $q \in F$ such that $0 < d(p, q) < r$. \square

Closures. For any subset E of a metric space X , we define the *closure* \bar{E} of E to be the set of all $p \in X$ such that $p \in E$ or p is a limit point of E (or both). That is, $\bar{E} := E \cup E'$ where E' is the set of all limit points of E in X . Clearly if $F \supseteq E$ then $F' \supseteq E'$ and thus $\bar{F} \supseteq \bar{E}$.

Theorem. [Simmons, page 68 and Exercise 6 on page 69] *For any subset E of a metric space X ,*

i) \bar{E} is closed.

ii) $E = \bar{E}$ if and only if E is closed.

iii) $\bar{E} \subseteq F$ for every closed set $F \subseteq X$ such that $F \supseteq E$.

[by (a) and (c), \bar{E} is the *smallest* closed subset of X that contains E , and the intersection of all closed $F \supseteq E$. NB this is a topological notion.]

Proof: (i) We must construct, for each $p \in X$ with $p \notin \bar{E}$, an open ball centered at p that is disjoint from \bar{E} . Since p is not a limit point of E , there exists $r > 0$ such that E contains no point q with $d(p, q) < r$ — note that we need not impose the usual constraint $q \neq p$, because we already assumed $p \notin \bar{E}$, and $\bar{E} \supseteq E$. Thus $B_r(p)$ is disjoint from E . We claim that it is also disjoint from E' . Indeed, suppose $q \in B_r(p)$. Since $B_r(p)$ is open, there exists $h > 0$ such that $B_h(q) \subseteq B_r(p)$. Thus $B_h(q)$ is disjoint from E , and q is not a limit point of E , as claimed. We conclude that $B_r(p)$ is disjoint from $E \cup E' = \bar{E}$, as desired.

(ii) (\Rightarrow) if $E = \bar{E}$ then E is closed by (i).

(\Leftarrow) If E is closed then we have seen $E' \subseteq E$, so $\bar{E} = E \cup E' = E$, as claimed.

(iii) We saw that if $F \supseteq E$ then $\bar{F} \supseteq \bar{E}$. But if F is closed then $\bar{F} = F$ by (ii). Thus $F \supseteq \bar{E}$. \square

In particular $\bar{B}_r(p) \supseteq \overline{B_r(p)}$ for all $r > 0$ (since $\bar{B}_r(p)$ is an example of a closed set that contains $B_r(p)$). In \mathbf{R}^n it is always true that $\bar{B}_r(p) = \overline{B_r(p)}$, but in some metric spaces $\bar{B}_r(p)$ may be strictly larger than $\overline{B_r(p)}$ for some p, r ; do you see how this can happen?

Going back to $X = \mathbf{R}$, we have:

Theorem. *Let $E \subset \mathbf{R}$ be a nonempty set bounded above. Then $\sup E \in \bar{E}$. In particular if E is closed then $E \ni \sup E$.*

Proof: Let $y = \sup E$. We prove that $y \in \bar{E}$ by contradiction. Assume that $y \notin \bar{E}$. Since \bar{E} is closed, there would then exist $h > 0$ such that $B_h(y)$ is disjoint from \bar{E} , and thus *a fortiori* from E . But then $y - h$ would be an upper bound on E strictly smaller than y . This is a contradiction, and we conclude that $y \in \bar{E}$. \square

This result will be fundamental to our rigorous development of the differential calculus.