Math 25b: Honors Linear Algebra and Real Analysis II

Homework Assignment #9 (4 April 2014):
Inverse and implicit functions

PROCEED FORMALLY, *verb phrase*: Manipulate symbols by the rules without any hint of their true meaning (popular in pure math courses).[1]

Inverse and implicit real-valued functions:

1. Solve Exercise 1.4 on page 171 of the Edwards text. (Edwards should have included the assumption $\partial G/\partial y \neq 0$ which is needed for the formula to make sense, and as we know is also needed to apply the implicit function theorem. As this Exercise suggests, an implicit function defined by a $\mathcal{C}^2$ equation is itself $\mathcal{C}^2$ once the condition $\partial G/\partial y \neq 0$ is satisfied. Higher derivatives work in the same way, but — as this Exercise also suggests — you don't want to derive the general formulas for d^3y/dx^3 and beyond. Like Edwards (see the brief Section 5), we shall punt on proving the $\mathcal{C}^k$ versions of the implicit and inverse function theorems, whose proofs are somewhat tedious and messy, and introduce no fundamentally new ideas.)

2.–3. Solve Exercise 1.5 on page 171, and Exercise 1.10 on pages 171–172.

4. i) Prove that there exists a differentiable real-valued function $y(x)$ in some neighborhood of $x = 0$ such that

$$x = e^{-y(x)}y(x)$$

for all x in that neighborhood.

ii) Show that $y(x)$ is a solution of the differential equation $xy' = y/(1-y)$, and conclude that y is a $\mathcal{C}^\infty$ (infinitely differentiable) function of x near the origin.

iii) Use that differential equation to show that the coefficients a_n of the Taylor expansion $y(x) = \sum_{n=1}^\infty a_n x^n$ of $y(x)$ about $x = 0$ satisfy the recurrence[2]

$$a_n = \frac{1}{n-1}\sum_{k=1}^{n-1} k a_k a_{n-k}$$

for $n > 1$; since also $a_1 = 1$ (why?) this inductively determines all the a_n. Compute a_2 through a_6. (Check: $a_6 = 54/5$.) Can you guess a formula for a_n? Is your guess confirmed by the value of a_7?

[I refrain from asking: iv) prove your guess. I know several "elementary" proofs of this, but none that is suitable for a homework problem.]

Inverse and implicit functions in higher dimensions:

5.–8. Solve Exercises 3.1, 3.8, 3.12, and 3.13 of the Edwards text (pages 194 and 195). For 3.13, it will help to recall the formula for the inverse of a 2×2 matrix. Also check that the formulas of 3.13 hold for the inverse function you found in 3.1 (whose similarity with the formula for the real and imaginary parts of $1/(x+iy)$ may be helpful).

[1] *Definitions of Terms Commonly Used in Higher Math*, R. Glover et al.; cf. also Prob. 4.

[2] Note that while we don't yet(?) know that convergent power series can be differentiated termwise, we *do* know that *Taylor series* can be differentiated termwise, because the x^k coefficient is *defined* in terms of the k-th derivative at zero.

9. Suppose A_0 is a diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$ such that $\lambda_1 \neq \lambda_j$ for all $j > 1$, so that λ_1 is an eigenvalue with multiplicity 1 (both geometric and algebraic multiplicity) and the first unit vector e_1 is an eigenvector. Prove that there exist $\mathcal{C}^1$ functions λ, v from a neighborhood U of A_0 *in the n^2-dimensional space of real $n \times n$ matrices* to $\mathbf{R}$ and $\mathbf{R}^n$ respectively, such that: $\lambda(A_0) = \lambda_1$, $v(A_0) = e_1$, and for all $A \in U$ we have $Av(A) = \lambda(A)v(A)$ and the first coordinate of $v(A)$ is 1. [Note that the last condition means that $v(A) - e_1$ is in the $n-1$ dimensional space orthogonal to e_1, so in effect we're looking for a map from $\mathbf{R}^{n^2}$ to $\mathbf{R}^{1+(n-1)} = \mathbf{R}^n$, determined by n conditions which are the coordinates of $Av(A) = \lambda(A)v(A)$.] Give a formula for the directional derivative of $\lambda(A)$ in the direction of an arbitrary $n \times n$ matrix M.

[The last problem is the beginning of "perturbation theory" for eigenvalues and eigenvectors, which is an important technique in quantum mechanics and elsewhere.]

This problem set is due Friday, April 11, at 5PM.