Math 25b: Honors Linear Algebra and Real Analysis II

Homework Assignment #8 (28 March 2014):
Taylor series in one and more variables; multivariate critical points

As soon as I get into class, I'm fighting off a swarm
Of positive-definite non-degenerate symmetric bilinear forms!
—from a somewhat redundantly titled patter-song in *Les Phys* (P. Dong, 2001)

Two approaches to the formula

$$\binom{k}{j_1, j_2, \ldots, j_n} = \frac{k!}{j_1!\, j_2! \cdots j_n!} \qquad (*)$$

for a multinomial coefficient (the coefficient of $x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$ in the expansion of the k-th power of a multinomial $x_1 + x_2 + \cdots + x_n$, necessarily with $j = k_1 + k_2 + \cdots + k_n$):

1. Solve Exercise 7.2 in Edwards (page 140).

2. We outline a proof of (*) by induction on k. The case $k = 0$ is clear: the only possibility for the j_i is $j_1 = j_2 = \cdots = j_n = 0$, and then both sides of (*) equal 1. For $k \geq 1$, prove that each side of (*) is the sum of the n terms obtained by replacing k by $k-1$ and one of the j_i by $j_i - 1$. [If some $j_i = 0$ then the term with j_i replaced by -1 is omitted. For the left-hand side, write

$$(x_1 + x_2 + \cdots + x_n)^k = (x_1 + x_2 + \cdots + x_n)\,(x_1 + x_2 + \cdots + x_n)^{k-1}$$

and compare $x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$ coefficients. For the right-hand side, use the recurrence $r! = r(r-1)!$. For both, it may help to orient yourself by starting with the $n = 2$ case $\binom{k}{j_1, j_2} = \binom{k-1}{j_1 - 1, j_2} + \binom{k-1}{j_1, j_2 - 1}$, usually written $\binom{k}{j_1} = \binom{k-1}{j_1 - 1} + \binom{k-1}{j_1}$.]

Some mostly computational problems with Taylor series in one, two, or three variables:

3.–7. Solve problems 7.4, 7.5, 7.6, 7.7, and 7.9 in Edwards (page 141).
[Note that you don't actually need to know what $\tan^{-1}$ and sin are. But I'm sure that you actually do know, so you can check the formulas in 7.5: for $\tan^{-1}$, the first derivative is $1/(1+x^2)$, and instead of computing further derivatives use geometric series together with Theorem 7.4; for $\sin^2(x)$, use $\frac{d}{dx}\sin(x) = \cos x$ and $\frac{d}{dx}\cos(x) = -\sin x$ to get the Taylor series for the sine and cosine, and then either proceed as in 7.8 or 7.9, or use the trig formula $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ which lets you obtain the entire Taylor series for $\sin^2 x$.]

8. Recall that the AM-GM inequality[1] for three variables says that (for $m > 0$) if x, y, and $z = 3m - x - y$ are all positive then $xyz \leq m^3$, with equality if and only if $x = y = z = m$. Thus $(x, y) = (m, m)$ must be a local maximum of the function $f(x, y) = xy(3m - x - y)$. Verify this by computing the quadratic form at that point and showing that it is negative-definite.

[1] AM = arithmetic mean; GM = geometric mean. If the arithmetic mean of x, y, z is m then $x + y + z = 3m$. The geometric mean is $\sqrt[3]{xyz}$, so AM$\geq$GM says $xyz \leq m^3$, etc.

9. Show that $q(x, y, z) = 2x^2 + 5y^2 + 2z^2 + 2xz$ is positive-definite by
   i) Solving the characteristic equation;
   ii) Verifying directly that $q(x, y, z) \geq 0$ with equality only when $x = y = z = 0$.

NB Problem 9 is not quite the same as Exercise 8.2 in Edwards (page 158).

Finally, about critical points whose quadratic forms fall through the cracks of Theorem 7.5 (page 138):

10. i) Suppose f is a $\mathcal{C}^3$ function in a neighborhood of the critical point $\mathbf{a}$ in $\mathbf{R}^n$ ($n \geq 2$) and that the quadratic form $q(\mathbf{h})$ is positive semidefinite but not zero. Prove that f cannot have a local maximum at $\mathbf{a}$. Give examples showing that f might, but does not have to, have a local minimum at $\mathbf{a}$. [Hint: Take $n = 2$ and $\mathbf{a} = (0, 0)$, and try functions of the form $f(x, y) = x^2 + g(y)$ for suitable functions g of one variable.]
    ii) Suppose instead that the quadratic form is zero. Give examples showing that each of the three possibilities might occur: f could have a local maximum, a local minimum, or no local extremum at $\mathbf{a}$.

This problem set is due Friday, April 4, at 5PM.