Math 25b: Honors Linear Algebra and Real Analysis II

Homework Assignment #7 (24 March 2014):
The Laplacian; single-variable Taylor series

> Mathematicians have long regarded it as demeaning to work on problems related to elementary geometry in two or three dimensions, in spite of the fact that it is precisely this sort of mathematics which is of practical value. — B. Grünbaum and G.C. Shephard, in *Handbook of Applied Mathematics*

[To be sure that's an overstatement, because even the most utilitarian sort of "practical value" can require higher-dimensional mathematical spaces (e.g. resource allocation via linear inequalities in many variables), and conversely some of the classics of elementary geometry have mainly esthetic rather than "practical" value (e.g. Brahmagupta's formula for the area of a cyclic quadrilateral). Still, mathematics in dimension ≤ 3 does offer more immediate connections with the Real World$^{\text{TM}}$, and also with our intuition, which we can then build on to go beyond dimension 3.]

This is again an abbreviated problem set because it covers material from only two classes, not the usual three. Most of these problems are motivated by the *Laplacian* ∇^2, which takes a twice-continuously-differentiable function f on $\mathbf{R}^n$ to $\nabla^2 f := \sum_{k=1}^n D_k^2 f = \sum_{k=1}^n \partial^2 f/\partial x_k^2$. This "second-order differential operator" is ubiquitous in physics, and often appears in other mathematical contexts.[1]

1. (cf. problems 3.6 and 3.7 in Edwards, page 88)
   If $g(u,v) = f(Au+Bv, Cu+Dv)$, where A, B, C, D are constants, show that

$$\nabla^2 g = (A^2+B^2)\frac{\partial^2 f}{\partial x^2} + 2(AC+BD)\frac{\partial^2 f}{\partial x\,\partial y} + (C^2+D^2)\frac{\partial^2 f}{\partial y^2}.$$

   In particular $\nabla^2 g = \nabla^2 f$ if the matrix $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ of our linear change of variables is orthogonal. For general M, give a description of the coefficients $A^2+B^2, 2(AC+BD), C^2+D^2$ in terms of M that suggests a generalization to higher dimensions.

This suggests one reason for the ubiquity of ∇^2: though defined in terms of coordinates, ∇^2 depends only on the inner-product structure of $\mathbf{R}^n$ and is not affected by *orthogonal* coordinate changes.

2. Solve Exercises 3.9 and 3.12 on page 89, which give ∇^2 in polar coordinates and use this to show that the real and imaginary parts of $(x+iy)^n$ satisfy the Laplace equation $\nabla^2 f = 0$ (see page 81). Check this last result directly for $n \leq 3$ by

[1] The "second" in "second-order" means that ∇^2 contains second partial derivatives but no higher. The "mixed partial" $D_x D_y$ also counts as a second mixed partial for this purpose.

Warning: In some contexts the Laplacian is defined to be what we call $-\nabla^2$. We shall never use this definition in Math 25b, but if you encounter "Laplacian" elsewhere, be sure to find out which sign is intended!

writing these real and imaginary parts in terms of x, y and comparing the second partial derivatives with respect to x and y (only the $n = 3$ case should take more than ϵ work).

A twice-continuously-differentiable function such as $\mathrm{Re}(x+iy)^n$ or $\mathrm{Im}(x+iy)^n$ that satisfies $\nabla^2 f = 0$ is said to be "harmonic". See also Problem 7.

Our next Edwards problem will require solving a differential equation, which we haven't covered yet. Hence the following preparatory exercise:

3. Suppose f, g are real-valued differentiable functions on an interval $I \subseteq \mathbf{R}$, and g is never zero on I. If f satisfies the differential equation $g(x)f'(x) = g'(x)f(x)$, show that there exists a constant c such that $f(x) = cg(x)$ for all $x \in I$. [Hint: we've shown that if $h : I \to \mathbf{R}$ is differentiable and $h'(x) = 0$ for all $x \in I$ then h is constant.]

4. Now solve Exercise 3.11 on page 89. In part (a), $n \leq 2$ is allowed as well, and g is assumed to be a twice-differentiable function from $(0, \infty)$ to $\mathbf{R}$.

The case $n = 3$ accounts for the importance of the potential functions $C/|\mathbf{x}|$ in classical physics (this is the form of both the gravitational potential of a point mass and the electrostatic potential of a point charge). For $n = 2$ we get $\log|\mathbf{x}|$ in place of $|\mathbf{x}|^{2-n}$, which explains why logarithmic potentials arise in 2-dimensional physics, as well as 3-dimensional models that are "effectively 2-dimensional" such as a charged infinite conducting wire. I didn't ask you to solve this $n = 2$ case because we haven't yet developed the logarithm function in Math 25.
Some applications of Taylor series of one variable:

5–6. Solve Exercises 6.7 and 6.9 in Edwards, page 128. For 6.9, give the answer for any function of the form $x^m(1-x)^n$ where m, n are integers greater than 1. (Here x is an arbitrary real number, not just $0 < x < 1$ as in some examples we've done in class.)

7. Suppose S is a nonempty open subset of $\mathbf{R}^n$ and $f : S \to \mathbf{R}$ is a function that is twice continuously differentiable. Suppose further that f has a local maximum or minimum at some $\mathbf{x}_0 \in S$. We know already that the first partial derivatives $D_j f$ must all vanish at $\mathbf{x} = \mathbf{x}_0$. Prove that if f is harmonic then the second partials $D_j D_k f$ all vanish at $(x, y) = (x_0, y_0)$ too. [Hint: you need only work one or two variables at a time. Once you've dealt with $D_j^2 f$, use Problem 1 to deal with the mixed partials $D_j D_k f$ for $j \neq k$.]

Much more is known: a harmonic function on an open set in $\mathbf{R}^n$ cannot have a local maximum or minimum at all unless it is constant. This is known as the "maximum principle" for harmonic functions. A consequence is that one cannot trap a particle at a point using just a static electromagnetic field (and whatever gravitational force may be present): there's always a path that deacreases the potential energy.

This problem set is due Friday, March 28, at 5PM.