Math 25b: Honors Linear Algebra and Real Analysis II

Homework Assignment #5 (28 February 2014):
Differentiation cont'd

Similarly, *adv.*: At least one line of the proof of this case is the same as before.[1]

More about the basics of derivatives:

0. If you haven't done it yet, solve Problem 1.1 of Edwards Chapter II (page 61), a.k.a. Problem 6 in the previous problem set.

1. Recall that $\mathbf{R}^*$ is the subset of $\mathbf{R}$ consisting of all real numbers other than 0. Define $f : \mathbf{R}^* \to \mathbf{R}$ by $f(x) = 1/x$. Prove from the definition (or an equivalent definition obtained in class) that f is differentiable at every $a \in \mathbf{R}^*$ and that its derivative is given by the formula $f'(a) = -1/a^2$.

2. Suppose $G \subset \mathbf{R}$ and that $g : G \to \mathbf{R}$ is a differentiable function such that $g(x) \neq 0$ for all $x \in G$. Thus there is a function $1/g : G \to \mathbf{R}$. Deduce from Problem 1 (and results covered in class) that g is differentiable and that its derivative is $-g'/g^2$ [i.e., for all $a \in G$ the derivative of $1/g$ at a is $-g'(a)/(g(a))^2$]. Deduce further that if $f : G \to \mathbf{R}$ is any differentiable function on G then f/g is differentiable and its derivative equals $(f'g - fg')/g^2$.

3. What is wrong with the following proof for the derivative of f/g? Let $h = f/g$, so that $f = gh$. By the formula for differentiating a product we have $f' = (gh)' = gh' + g'h$. Hence $gh' = f' - g'h = f' - g'(f/g) = (f'g - fg')/g$, so dividing by g we obtain $h' = (f'g - fg')/g^2$ as claimed.

A couple of problems using higher derivatives:

4. i) If $f, g : [a, b] \to \mathbf{R}$ are thrice[2] differentiable at $x \in [a, b]$, prove that so is their product fg, and find $(fg)'''(x)$.
   ii) Generalize.

5. i) If $f : [a, b] \to \mathbf{R}$ is twice differentiable and satisfies $f(a) = f(b) = 0$ and also $f(c) = 0$ for some c with $a < c < b$, prove that there exists $x \in (a, b)$ such that $f''(x) = 0$. Generalize.
   ii) If $f : [-1, 1] \to \mathbf{R}$ is twice differentiable with $f(-1) = f(1) = 0$ but $f(0) = 1$, prove that there exists $x \in (-1, 1)$ such that $f''(x) = -2$.

[1] *Definitions of Terms Commonly Used in Higher Math*, R. Glover et al. Note that this does <u>not</u> define an equivalence relation.

[2] once : twice : thrice :: 1 : 2 : 3. Look it up if you don't believe me. As far as I know the sequence "once, twice, thrice" has no fourth term in English (though it does have a zeroth term of sorts in "never").

The following two problems concern derivatives of matrix-valued functions. Recall (from Math 25a) that $\mathcal{L}(\mathbf{R}^m, \mathbf{R}^n)$ is the vector space of linear transformations from $\mathbf{R}^m$ to $\mathbf{R}^n$, which can be identified with the vector space of $m \times n$ matrices. In particular we already know what it means for a function f from some $G \subset \mathbf{R}$ to $\mathcal{L}(\mathbf{R}^m, \mathbf{R}^n)$ to be differentiable: such a function is just an $m \times n$ array of functions f_{ij}, and f is differentiable iff each f_{ij} is, in which case f' is the array of derivatives f'_{ij}.

6. If both $f : G \to \mathcal{L}(\mathbf{R}^n, \mathbf{R}^p)$ and $g : G \to \mathcal{L}(\mathbf{R}^m, \mathbf{R}^n)$ are differentiable, prove that so is the function $fg : G \to \mathcal{L}(\mathbf{R}^m, \mathbf{R}^p)$ taking each $x \in G$ to the matrix product of $f(x)$ with $g(x)$. Prove that the formula $(fg)' = fg' + f'g$ holds in this setting too. (We've in effect seen the special cases $m = n = 1$ and $m = p = 1$ already.)

7. Suppose $f : G \to \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$ is differentiable, and that $\det(f(x)) \neq 0$ for all $x \in G$. Then there is a function $f^{-1} : G \to \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$ taking each $x \in G$ to the matrix inverse of $f(x)$.
   i) Prove that f^{-1} is differentiable at all $x \in G$. [Hint: remember the "Cramer's rule" formula for the inverse of an invertible matrix in terms of its determinant and cofactors.]
   ii) Prove that the derivative of this function f^{-1} at any $x \in G$ equals $-f^{-1}(x)f'(x)f^{-1}(x)$. [Hint: forget Cramer's rule. Instead look earlier in this problem set.]

Finally, about the derivatives and the Chain Rule for functions of several variables:

8. Solve Problem 2.3 on page 75 of the Edwards text [$xy^2/(x^2 + y^2)$ is not differentiable at $(x, y) = (0, 0)$ even though all directional derivatives exist].

9. Define $f : \mathbf{R}^2 \to \mathbf{R}$ by $f(x, y) = xy$. Prove from the definition (or an equivalent definition obtained in class) that f is differentiable at every $(a, b) \in \mathbf{R}^2$ and that its differential at (a, b) is the linear transformation taking any $(u, v) \in \mathbf{R}^2$ to $av + bu$. [You can, but might not want to, use the Hint that Edwards gives for Problem 2.2 on page 75.]

10. Use this to give a different proof of the formula for the derivative of a product of two differentiable functions $f, g : G \to \mathbf{R}$ on some $G \subset \mathbf{R}$ by writing fg as a composite of two differentiable functions $G \to \mathbf{R}^2 \to \mathbf{R}$.

The problem set is due Friday, March 7, at 5PM.