Math 25b: Honors Linear Algebra and Real Analysis II

Homework Assignment #4 (21 February 2014):
Metric Topology IV; introduction to differential calculus

> [...] what are these Fluxions [i.e. derivatives]? The Velocities of evanescent Increments? And what are these same evanescent Increments? They are neither finite Quantities nor Quantities infinitely small, nor yet nothing. May we not call them the Ghosts of departed Quantities?
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> —George Berkeley (1685–1753), in *The Analyst* (1734)
>
> We define $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ if this limit [taken over $h \neq 0$] exists [...]
>
> —Edwards (1973), page 57, after Cauchy (1789–1857) et al.

The following problem uses a basic property of compact sets that we somehow haven't used yet:

1. Suppose that $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$ and that X is any metric space.[1] A function $f : X \to \mathbf{F}$ is said to have *compact support* if X has a compact subset K such that $f(x) = 0$ for all $x \notin K$. Prove that the functions of compact support on X form a vector space. [Hint: you'll want to solve part of Problem 3 in Simmons page 114.]

(In general the "support" of a function $X \to \mathbf{F}$ is the set of all $x \in X$ such that $f(x) \neq 0$.)

Another use of compactness:

2. Suppose X is a nonempty compact metric space, and let $f : X \to X$ be any continuous function such that $d(f(x), f(x')) < d(x, x')$ for all distinct $x, x' \in X$. Prove that f has a fixed point, i.e. that there exists $x \in X$ such that $f(x) = x$. (Hint: the method Simmons uses for the Lemma on page 338 doesn't work here! Instead, show that $\delta(x) = d(x, f(x))$ defines a continuous function $\delta : X \to [0, \infty)$, and use the fact that a real-valued continuous function on a compact set attains its infimum and supremum.)

Can you give a counterexample where X is complete but not compact and $f : X \to X$ is a continuous function that satisfies $d(f(x), f(x')) < d(x, x')$ for all distinct $x, x' \in X$ but has no fixed point?

More about completeness:

3. Let $f : X \to Y$ be a continuous function between metric spaces.
   i) Prove that if f is uniformly continuous then the image of a Cauchy sequence is Cauchy.
   ii) For $X, Y = (0, \infty)$ find a continuous $f : X \to Y$ and a Cauchy sequence $\{x_n\}$ such that $\{f(x_n)\}$ is not Cauchy. [Thus f cannot be uniformly continuous, and the uniformity hypothesis cannot be dropped from the first part. Your f will probably have a continuous inverse function, i.e. a function $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are the identity functions on Y and X respectively; such an f also shows that "Cauchy sequence" is <u>not</u> a topological notion — do you see why?]

[1] A topological space would suffice.

4. Let Y be a metric space, X an arbitrary set, and $\{f_n\}$ a sequence of functions from X to Y. We saw that if the f_n are bounded then f_n approaches a function $f : X \to Y$ in the $\mathcal{B}(X, Y)$ metric if and only if $f_n \to f$ uniformly. What should it mean for a sequence $\{f_n\}$ to be "uniformly Cauchy"? Prove that if Y is complete and X is a metric space then a uniformly Cauchy sequence of continuous functions from X to Y converges uniformly to a continuous function.

Another consequence of our analysis of polynomial functions on $\mathbf{C}$:

5. Suppose $f \in \mathbf{C}[z]$ is a nonconstant polynomial. Prove that $|f|$ has no local maximum, i.e. that for every $z \in \mathbf{C}$ and $\epsilon > 0$ there exists z' such that $|z - z'| < \epsilon$ and $|f(z')| > |f(z)|$. Deduce that if f is bounded on some nonempty subset $K \subset \mathbf{C}$ then every $z_1 \in K$ such that $|f(z_1)| = \sup_{z \in K} |f(z)|$ is on the boundary of K. (In particular this is true for K compact, in which case we know that at least one such z_1 exists.) Must the same be true with "supremum" replaced by "infimum"?

Finally, some differential calculus problems from Edwards:[2]

6.–10. Solve problems 1.1, 1.5, 1.6, 1.9, and 1.12 in Edwards Chapter II (pages 61–63). For 1.12, a and b are positive constants (and the punchline of 1.12c does not require the third coordinate).

The problem set is due Friday, February 28, at 5PM.

[2]Edwards uses $\mathscr{R}$ for what we call $\mathbf{R}$ and handwrite as $\mathbb{R}$. Please don't use $\mathscr{R}$ in your writeups...