Math 25b: Honors Linear Algebra and Real Analysis II

Homework Assignment #3 (14 February 2014):
Metric Topology III: uniform continuity, compactness, and completeness

"There goes the open ball."

—Mangled punchline of a hoary "$\epsilon \to 0$" joke

Due to the Feb. 17 holiday, this problem set covers only two class meetings; therefore it consists of only six problems.

Uniform vs. non-uniform continuity:

1. Let P be a polynomial with real coefficients, considered as a function from $\mathbf{R}$ to $\mathbf{R}$ (with the usual metric on both the source and target $\mathbf{R}$). We have seen in the previous problem set that P is continuous.
   i) Prove that if $\deg(P) \le 1$ then P is uniformly continuous.
   ii) Prove that if $\deg(P) > 1$ then P is <u>not</u> uniformly continuous. [Hint: show that if any function $f : \mathbf{R} \to \mathbf{R}$ is uniformly continuous then there exists $\delta > 0$ such that $|f(n\delta) - f(0)| \le n$ for each $n = 1, 2, 3, \ldots$.]

2. Let m and n be any positive integers. Prove that every linear transformation from $\mathbf{R}^m$ to $\mathbf{R}^n$ is *uniformly* continuous. [You may use either the sup metric or the Euclidean metric. You'll first have to prove the function is continuous at $\mathbf{0}$; use the matrix form of a linear transformation.]

We noted that once we've shown that the product of two compact spaces is compact then by induction the same is true for any <u>finite</u> product of compact spaces. But (using the sup metric) it doesn't work for a countably infinite product...

3. Let X be the product of a countable infinite of copies of the unit interval $[0, 1]$, or equivalently of sequences $p = (p_1, p_2, p_3, \ldots)$ with each $p_n \in [0, 1]$. Then $d(p, q) = \sup_n |p_n - q_n|$ defines a metric on X. Prove that X is not totally bounded (first find a sequence $x_1, x_2, \ldots$ with each $x_m \in X$ such that $d(x_m, x_n) = 1$ for all $m \ne n$), and therefore not compact.

A non-convergent Cauchy sequence in a function space:

4. As in Problem 6 of the previous problem set, define a metric

$$d_1(f, g) := \int_{-1}^{1} |f(x) - g(x)| \, dx$$

on the space $\mathcal{C}([-1, 1], \mathbf{R})$ of (bounded) continuous functions $f : [-1, 1] \to \mathbf{R}$.[1] Let f_n be the function given by

$$f_n(x) = \begin{cases} nx, & \text{if } |nx| \le 1; \\ 1, & \text{if } nx \ge 1; \\ -1, & \text{if } nx \le -1. \end{cases}$$

[1] We previously used $[0, 1]$ but for the present purpose $[-1, 1]$ is more convenient; and we do not need complex-valued functions so we content ourselves with $\mathcal{C}([-1, 1], \mathbf{R})$ rather than $\mathcal{C}([-1, 1], \mathbf{C})$.

(Note that when $x = \pm 1/n$ two clauses apply but both give ± 1, so the definition is consistent.)

i) Show that each f_n is continuous, and thus contained in $\mathcal{C}([-1,1],\mathbf{R})$.

ii) Calculate that $d_1(f_m, f_n) = |1/m - 1/n|$, and deduce that $\{f_n\}$ is a Cauchy sequence in $\mathcal{C}([-1,1],\mathbf{C})$.

Intuitively the reason that this sequence has no limit in $\mathcal{C}([-1,1],\mathbf{C})$ is that the limit should be the function that's -1 for $x < 0$, zero for $x = 0$, and $+1$ for $x > 0$, and this function is not continuous at $x = 0$. Here's a proof:

5. Let f be any continuous function from $[-1,1]$ to $\mathbf{R}$. There exists δ such that $|x| \leq \delta \Rightarrow |f(x) - f(0)| < \frac{1}{4}$ (why?). Show that once $1/n < \delta/2$ the distance from f to f_n is at least $\delta/4$ (due either to the integral over $-\delta \leq x \leq -\delta/2$ or to the integral over $\delta/2 \leq x \leq \delta$). Explain why this shows that f is <u>not</u> a limit of $\{f_n\}$, and thus that $\{f_n\}$ is not convergent in $\mathcal{C}([-1,1],\mathbf{R})$.

An application of total boundedness in $\mathbf{R}$ and $\mathbf{R}^2$:

6. i) Fix a real number x, and let for each $n = 1, 2, 3, \ldots$ let p_n be the fractional part of nx. (The fractional part of a real number r is the unique real number $f \in [0,1)$ such that $r - f$ is an integer. For example, if $x = e$ then $p_1 = e - 2 = 0.71828\ldots$, $p_2 = 2e - 5 = 0.43656\ldots$, and $p_{100} = 100e - 271 = 0.82818\ldots$; if some nx is an integer then we take $p_n = 0$.) Thus $\{p_n\}$ a sequence in the semiopen interval $[0,1)$. This interval, regarded as a subspace of $\mathbf{R}$ (with the usual metric structure), is totally bounded. Therefore the sequence $\{p_n\}$ has a Cauchy subsequence. Use this to deduce that for every ϵ there exist integers a, b with $a > 0$ such that $|ax - b| < \epsilon$.

   ii) Now let x, y be real numbers and let p_n be the point in $\mathbf{R}^2$ whose coordinates are the fractional parts of nx and ny. This is a sequence in $[0,1)^2$, a subspace of $\mathbf{R}^2$ (use the standard metric on $\mathbf{R}$ and the sup metric on $\mathbf{R}^2$). Use the total boundedness of $[0,1)^2$ to deduce that for every ϵ there exist integers a, b, c such that $|ax - b| < \epsilon$ and $|ay - c| < \epsilon$.

Yes, the same argument shows that for real $x_1, \ldots, x_k$ and any ϵ there are integers $a > 0$ and $b_1, \ldots, b_k$ such that $|ax_i - b_i| < \epsilon$ for each $i = 1, 2, \ldots, k$.

The problem set is due Friday, February 21, at 5PM.