

Math 25b: Honors Linear Algebra and Real Analysis II

Homework Assignment #2 (7 February 2014):
Metric Topology II: continuity, sequences, and compactness

Sketch of a proof n . I couldn't verify all the details, so I'll break it down into the parts I couldn't prove.¹

This problem set is due Friday, **February 14, at 5 pm** in the **CA mailboxes** on the second floor of the Science Center (which are now available). Please write out the problems corresponding to each CA separately (using \TeX is strongly encouraged!).

Please avoid merely "sketching" (as defined in the above quote) a proof. In all problem sets, you may use any result proved in class; and you may use the result in one problem (or problem part) to solve another, even if you have not proved the first one, unless explicitly told not to (e.g. in Problem 9), and unless this becomes circular. [EXCEPTION: when problem B is clearly a generalization of A, don't use B to solve A unless you've solved B!] The problems are generally not in order of difficulty.

For Sadik Shahidain

More about the topology of \mathbf{R} , and a connection with continuity:

1. Prove that a subset of \mathbf{R} is open if and only if it is a union of a set of open intervals (x, y) with $x < y$ and x, y both rational. Deduce that the topology of \mathbf{R} (i.e. the collection of open subsets of \mathbf{R}) has the same cardinality as \mathbf{R} . [This was mentioned without proof in class, together with the reminder that the collection of all subsets of \mathbf{R} has larger cardinality.] Generalize to the topology of \mathbf{R}^n for any positive integer n .
2. i) Prove that the only subsets of \mathbf{R} that are simultaneously open and closed are \emptyset and \mathbf{R} . [Hint: Recall that a bounded and closed subset of \mathbf{R} contains its supremum and infimum.]
ii) Suppose X, Y are metric spaces, and that X has the discrete metric. Find all continuous maps from X to Y . Find all continuous maps from \mathbf{R} to X .
3. Let $\mathbf{F} = \mathbf{R}$ or \mathbf{C} with the standard metric. Prove that every polynomial $f(x)$ with coefficients in \mathbf{F} defines a continuous function from \mathbf{F} to \mathbf{F} . Prove that every rational $f(x)/g(x)$ (where f and g are polynomials) with coefficients in \mathbf{F} defines a continuous function from $g^{-1}(\mathbf{F}^*)$ to \mathbf{F} . [Recall that \mathbf{F}^* consists of the nonzero elements of \mathbf{F} .] Generalize to polynomials and rational functions in several real or complex variables. [Hint: you do not want to work out a direct ϵ - δ proof.]

For Tudor Giurgica-Tiron

Another formulation of convergence:

¹*Definitions of Terms Commonly Used in Higher Math*, R. Glover et al.

4. Fix a sequence $\{r_n\}$ of positive real numbers such that $r_1 > r_2 > r_3 > \dots$ and $r_n \rightarrow 0$. Let $\tilde{\mathbf{N}}$ be the metric space consisting of $1, 2, 3, \dots$ together with a symbol ∞ , with the distance function defined by

$$d(m, n) = |r_m - r_n|, \quad d(n, \infty) = d(\infty, n) = r_n, \quad d(\infty, \infty) = 0.$$

[In other words, d is defined so that the map $\rho : \tilde{\mathbf{N}} \rightarrow \mathbf{R}$ given by $n \mapsto r_n$, $\infty \mapsto 0$ is an isometry to $\rho(\tilde{\mathbf{N}})$.] Let \mathbf{N} be the subspace $\{1, 2, 3, \dots\}$ of $\tilde{\mathbf{N}}$, so $\tilde{\mathbf{N}}$ is the disjoint union of \mathbf{N} with $\{\infty\}$.

- i) Which subsets of \mathbf{N} are open?
 ii) Which subsets of $\tilde{\mathbf{N}}$ are open?
5. Let $\{s_n\}$ be a sequence in an arbitrary metric space X . Let $\sigma : \mathbf{N} \rightarrow X$ be the map that takes n to s_n . Show that $\{s_n\}$ converges if and only if σ extends to a continuous function $\tilde{\sigma} : \tilde{\mathbf{N}} \rightarrow X$ (that is, if and only if there exists a continuous $\tilde{\sigma} : \tilde{\mathbf{N}} \rightarrow X$ such that $\tilde{\sigma}(n) = \sigma(n)$ for all $n \in \mathbf{N}$), in which case $\tilde{\sigma}(\infty) = \lim_{n \rightarrow \infty} s_n$.

More about sequences and $\mathcal{C}(X, Y)$:

6. Prove that²

$$d_1(f, g) := \int_0^1 |f(x) - g(x)| dx$$

is a metric on the space $\mathcal{C}([0, 1], \mathbf{C})$ of (bounded) continuous functions $f : [0, 1] \rightarrow \mathbf{C}$ on the closed unit interval $[0, 1]$.

7. Let X be our metric space $\mathcal{C}([0, 1], \mathbf{R})$ of (bounded) continuous functions on $[0, 1]$ with $d_X(f, g) = \sup_{0 \leq x \leq 1} |f(x) - g(x)|$.
- i) Find an infinite set $S \subset X$ such that the restriction of d_X to S is the discrete metric. Could you do the same for the metric d_1 of the previous problem?
- ii) Let $\mathbf{0} \in X$ be the zero function. Is the closed unit ball $\bar{B}_1(\mathbf{0})$ in X compact? Why?

For Nat Mayer

8. Define sequences $\{f_n\}, \{g_n\}$ ($n = 1, 2, 3, \dots$) of functions from \mathbf{R} to \mathbf{R} by

$$f_n(x) = \frac{n}{x^2 + n^2}, \quad g_n(x) = \frac{n^2}{x^2 + n^2}.$$

- i) Do these sequences of functions f_n and g_n converge pointwise?
 ii) Do they converge uniformly on \mathbf{R} ? Explain.

More about compact metric spaces:

9. Show directly that a sequentially compact subset of a metric space is closed and totally bounded.
10. Say that a subset E of a metric space X is “totally bounded relative to X ” if, for each $\epsilon > 0$, there is a finite cover of E by ϵ -neighborhoods in X . Prove that E is totally bounded relative to X if and only if E is totally bounded. [That is, allowing centers of the ϵ -neighborhoods to be in a larger ambient metric space does not change the notion of total boundedness.]

²Yes, I know: we have yet to officially define \int_0^1 . For this problem, though, only the most basic facts are needed, such as the existence of $\int_0^1 F(x) dx$ for any (bounded) continuous function $F : [0, 1] \rightarrow \mathbf{R}$, and the fact that if $F, G : [0, 1] \rightarrow \mathbf{R}$ are continuous with $F(x) \leq G(x)$ for all $x \in [0, 1]$ then $\int_0^1 F(x) dx \leq \int_0^1 G(x) dx$. Only ϵ more is needed for the next problem.