

Math 25b: Honors Linear Algebra and Real Analysis

Homework Assignment #1 (31 January 2014):

Metric topology basics

“I’m sorry...”

“Don’t topologize.”

—Martin Gardner, *The Unexpected Hanging*

This problem set is due Friday, February 7, at 5PM. Please write out each of the three parts separately and submit them to the CA mailboxes on the second floor of the Science Center. If you choose to use \LaTeX to typeset your assignments (which is strongly encouraged), please print out and submit a hard copy.

For Nat Mayer

Definition and constructions of metric spaces:

1. For any set X define the *discrete metric* on X by $d(p, q) = 0$ if $p = q$ and $d(p, q) = 1$ if $p \neq q$. (See Simmons, page 51, Example 1.) Prove that this is indeed a metric. With this metric, which subsets of X are open? Which are closed? Which are dense?
2. [Simmons, page 58, Problem 1] Let (X, d) be a metric space. Define

$$d_0(x, y) := \frac{d(x, y)}{1 + d(x, y)}$$

for all $x, y \in X$.

- i) Prove that d_0 is also a metric on X .
 - ii) Prove that a subset of X is open under the metric d if and only if it is open under d_0 . [Thus (X, d) and (X, d_0) are the same as “topological spaces”, but generally not isometric (identical as metric spaces).]
 - iii) Show that the metric space (X, d_0) is always bounded, even though (X, d) might not be.
3. Which of the following defines a metric on \mathbf{R} ? Explain. (For a hint see Problem 10.)
 - i) $d_1(x, y) := (x - y)^2$
 - ii) $d_2(x, y) := |x^2 - y^2|$
 - iii) $d_3(x, y) := |x^3 - y^3|$
 - iv) $d_4(x, y) := \sqrt{|x - y|}$
 - v) $d_5(x, y) := |x - 2y|$
 - vi) $d_6(x, y) := |x - y|/(1 + |x - y|)$

For Tudor Giurgica-Tiron

Problems 4 and 5 concern isometries between metric spaces. Recall that an *isometry* between metric spaces X, Y is a bijection $i : X \rightarrow Y$ such that

$$d_Y(i(x_1), i(x_2)) = d_X(x_1, x_2)$$

for all $x_1, x_2 \in X$. If $X = Y$ we say i is an “isometry of X ”.

4. Prove that: the identity map on a metric space is an isometry; if $i : X \rightarrow Y$ is an isometry, then so is the inverse map $i^{-1} : Y \rightarrow X$; and if $i : X \rightarrow Y$ and $j : Y \rightarrow Z$ are isometries, then so is the composite map $j \circ i : X \rightarrow Z$. (This is tantamount to saying that the isometries of a metric space constitute a group under composition of functions.)
5. i) For $X = Y = \mathbf{R}$, prove that the function $i(x) = -x$ is an isometry, as is $j_a(x) = x + a$ for any $a \in \mathbf{R}$.
 ii) Prove that every isometry from \mathbf{R} to itself is either j_a or $i \circ j_a$ for some a .

Closures, etc.:

6. Let E be a subset of a metric space, and E' its set of limit points. Prove that E' is closed, and that E and \bar{E} have the same limit points. (Recall that \bar{E} , the *closure* of E , can be defined by $\bar{E} = E \cup E'$.) Is it true that E and E' have the same limit points for every E ?

For Sadik Shahidain

7. Let A_1, A_2, A_3, \dots be subsets of a metric space.
 i) If $B_n = \cup_{i=1}^n A_i$, prove that $\bar{B}_n = \cup_{i=1}^n \bar{A}_i$ (the closure of a finite union is the union of the closures).
 ii) If $B = \cup_{i=1}^{\infty} A_i$, prove that $\bar{B} \supseteq \cup_{i=1}^{\infty} \bar{A}_i$ (the closure of a countable union contains the union of the closures).
 iii) Give an example where this inclusion is proper (a.k.a. strict), that is, an example of a metric space with subsets A_i such that $\bar{B} \neq \cup_{i=1}^{\infty} \bar{A}_i$.

Continuous functions:

8. [Simmons, page 79, Problem 3]¹ Let X and Y be metric spaces and f, g continuous functions from X to Y . If $A \subset X$ is a subset such that $f(x) = g(x)$ for all $x \in A$, prove that $f(x) = g(x)$ for all x in the closure \bar{A} .
9. [Simmons, page 79, Problem 5] Suppose that X and Y are metric spaces and that X has the discrete metric. Prove that every function from X to Y is continuous.

Uniform continuity:

10. Let d, d', d'' be the following three metrics on \mathbf{R} :
 d is the standard metric $d(x, y) = |x - y|$;
 d' is the discrete metric; and
 d'' is the metric $d''(x, y) = |x^3 - y^3|$

The identity function $x \mapsto x$ on \mathbf{R} then gives rise to six functions

$$(\mathbf{R}, d) \rightleftharpoons (\mathbf{R}, d'), \quad (\mathbf{R}, d') \rightleftharpoons (\mathbf{R}, d''), \quad (\mathbf{R}, d'') \rightleftharpoons (\mathbf{R}, d).$$

- i) Which of these six functions are continuous?
 ii) Of those, which are uniformly continuous?

¹Simmons assumes that $A \neq \emptyset$, but as far as I can see this is not needed because the result is true (albeit vacuous) also if $A = \emptyset$.