Math 25b: Honors Linear Algebra and Real Analysis II

Homework Assignment #12 (25–30 April 2014):
Multiple integrals, cont'd; line integrals and Green's theorem

$$\int_{\partial D} \omega = \int_D d\omega$$

—Lord Kelvin, as communicated (1850) to George Stokes and greatly generalized thereafter

1-4. [**Sadik**] Edwards 4.1, 4.2, 4.3, 4.5 (page 243)

5-7. [**Nat**] Edwards 5.3, 5.13, 5.14 (pages 264-266)

8. [**Tudor**] Suppose $x_0 > 0$ and let $g : [0, x_0] \to \mathbf{R}$ be a decreasing $\mathcal{C}^1$ function. Let D be the plane region
$$\{(x, y) \in \mathbf{R}^2 \mid 0 \le x \le x_0,\ 0 \le y \le f(x)\}$$
whose boundary ∂D consists of the line segments from $(0, f(0))$ to $(0, 0)$ to $(0, x_0)$ and the graph of g. Prove Green's theorem $\int_D \omega = \int_{\partial D} d\omega$ for this region (where ω is any continuous differential form on an open set containing D) by adapting the technique we used in class (see Edwards, pages 311–312) to prove Green's for boxes).

It might help to warm up with the special case $x_0 = 1$, $g(x) = 1 - x$, which makes D an isosceles right triangle. This result, together with the technique of tiling and canceling shared boundary components (see Edwards, pages 316–317), gives an alternative route for proving the statement of Green's theorem (instead of verifying that all of its ingredients are invariant under invertible $\mathcal{C}^1$ changes of coordinate) for most cases that one is likely to need, such as the circle of the next (and final!) problem.

9. [**Tudor**] Let $U \subset \mathbf{R}^2$ be an open set containing the round disc $\{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 \le R^2\}$ for some $R > 0$, and let $F : U \to \mathbf{R}$ be any $\mathcal{C}^1$ function, with partial derivatives $F_x = \partial F/\partial x$, $F_y = \partial F/\partial y$. For $0 \le r \le R$, define
$$\alpha(r) = \frac{1}{2\pi} \int_0^{2\pi} F(r\cos\theta, r\sin\theta)\, d\theta;$$
that is, $\alpha(r)$ is the average of F over the circle of radius r.
i) Prove that α is $\mathcal{C}^1$ with
$$\alpha'(r) = \frac{1}{2\pi} \int_0^{2\pi} \big(\cos\theta \cdot F_x(r\cos\theta, r\sin\theta) + \sin\theta \cdot F_y(r\cos\theta, r\sin\theta)\big)\, d\theta,$$
and that for $r > 0$ this can be written as a line integral $\alpha'(r) = (2\pi r)^{-1} \int_{\gamma_r} \omega$, where ω is the differential form $-F_y\, dx + F_x\, dy$ on U, and γ_r is the path $(0, 2\pi) \to U$,

$\theta \mapsto (r\cos\theta, r\sin\theta)$.

ii) Suppose now that F is $\mathcal{C}^2$, so that ω is a $\mathcal{C}^1$ form. Deduce that

$$\alpha'(r) = \frac{1}{2\pi r} \iint_{x^2+y^2<r^2} (F_{xx} + F_{yy})\, dx\, dy,$$

and check directly that this formula is correct for $F(x, y) = x^2 + y^2$.

iii) Conclude that if F is harmonic (i.e. a $\mathcal{C}^2$ function F satisfying the differential equation $F_{xx} + F_{yy} = 0$) then $F(\vec{x})$ equals the average of F over any circle centered at x that bounds a disc contained in U. Recover the maximum principle for harmonic functions in the plane: a harmonic function on an open set in $\mathbf{R}^2$ cannot have a local maximum unless it is locally constant.

More generally, the Laplacian measures how $F(\vec{x})$ compares with its average over small circles around $\vec{x}$. Stokes' theorem for regions in $\mathbf{R}^n$ generalizes this, and the proof of the maximum principle, to arbitrary dimension n (where naturally circles are replaced by spheres).

This problem set is due Monday, May 5, at 5PM.