Math 25b: Honors Linear Algebra and Real Analysis II

Homework Assignment #1 (31 January 2014):
Metric topology basics

"I'm sorry…"

"Don't topologize."

—Martin Gardner, *The Unexpected Hanging*

Definition and constructions of metric spaces:

1. For any set X define the *discrete metric* on X by $d(p,q)=0$ if $p=q$ and $d(p,q)=1$ if $p\neq q$. (See Simmons, page 51, Example 1.) Prove that this is indeed a metric. With this metric, which subsets of X are open? Which are closed? Which are dense?

2. [Simmons, page 58, Problem 1] Let (X,d) be a metric space. Define

$$d_0(x,y) := \frac{d(x,y)}{1+d(x,y)}$$

for all $x,y\in X$.
i) Prove that d_0 is also a metric on X.
ii) Prove that a subset of X is open under the metric d if and only if it is open under d_0. [Thus (X,d) and (X,d_0) are the same as "topological spaces", but generally not isometric (identical as metric spaces).]
iii) Show that the metric space (X,d_0) is always bounded, even though (X,d) might not be.

3. Which of the following defines a metric on $\mathbf{R}$? Explain. (For a hint see Problem 10.)
i) $d_1(x,y) := (x-y)^2$
ii) $d_2(x,y) := |x^2-y^2|$
iii) $d_3(x,y) := |x^3-y^3|$
iv) $d_4(x,y) := \sqrt{|x-y|}$
v) $d_5(x,y) := |x-2y|$
vi) $d_6(x,y) := |x-y|/(1+|x-y|)$

Problems 4 and 5 concern isometries betwen metric spaces. Recall that an *isometry* between metric spaces X,Y is a bijection $i : X\to Y$ such that

$$d_Y(i(x_1), i(x_2)) = d_X(x_1,x_2)$$

for all $x_1,x_2\in X$. If $X=Y$ we say i is an "isometry of X".

4. Prove that: the identity map on a metric space is an isometry; if $i : X\to Y$ is an isometry, then so is the inverse map $i^{-1} : Y\to X$; and if $i : X\to Y$ and $j : Y\to Z$ are isometries, then so is the composite map $j\circ i : X\to Z$. (This is tantamount to saying that the isometries of a metric space constitute a group under composition of functions.)

5. i) For $X = Y = \mathbf{R}$, prove that the function $i(x) = -x$ is an isometry, as is $j_a(x) = x + a$ for any $a \in \mathbf{R}$.
ii) Prove that every isometry from $\mathbf{R}$ to itself is either j_a or $i \circ j_a$ for some a.

Closures, etc.:

6. Let E be a subset of a metric space, and E' its set of limit points. Prove that E' is closed, and that E and $\overline{E}$ have the same limit points. (Recall that $\overline{E}$, the *closure* of E, can be defined by $\overline{E} = E \cup E'$.) Is it true that E and E' have the same limit points for every E?

7. Let $A_1, A_2, A_3, \ldots$ be subsets of a metric space.
i) If $B_n = \cup_{i=1}^n A_i$, prove that $\overline{B}_n = \cup_{i=1}^n \bar{A}_i$ (the closure of a finite union is the union of the closures).
ii) If $B = \cup_{i=1}^\infty A_i$, prove that $\overline{B} \supseteq \cup_{i=1}^\infty \bar{A}_i$ (the closure of a countable union contains the union of the closures).
iii) Give an example where this inclusion is proper (a.k.a. strict), that is, an example of a metric space with subsets A_i such that $\overline{B} \neq \cup_{i=1}^\infty \bar{A}_i$.

Continuous functions:

8. [Simmons, page 79, Problem 3][1] Let X and Y be metric spaces and f, g continuous functions from X to Y. If $A \subset X$ is a subset such that $f(x) = g(x)$ for all $x \in A$, prove that $f(x) = g(x)$ for all x in the closure $\bar{A}$.

9. [Simmons, page 79, Problem 5] Suppose that X and Y are metric spaces and that X has the discrete metric. Prove that *every* function from X to Y is continuous.

Uniform continuity:

10. Let d, d', d'' be the following three metrics on $\mathbf{R}$:
d is the standard metric $d(x, y) = |x - y|$;
d' is the discrete metric; and
d'' is metric d_3 from Problem 3.
The identity function $x \mapsto x$ on $\mathbf{R}$ then gives rise to six functions

$$(\mathbf{R}, d) \rightleftarrows (\mathbf{R}, d'), \quad (\mathbf{R}, d') \rightleftarrows (\mathbf{R}, d''), \quad (\mathbf{R}, d'') \rightleftarrows (\mathbf{R}, d).$$

i) Which of these six functions are continuous?
ii) Of those, which are uniformly continuous?

The problem set is due Friday, February 7, at 5PM.

[1] Simmons assumes that $A \neq \emptyset$, but as far as I can see this is not needed because the result is true (albeit vacuous) also if $A = \emptyset$.