

## Math 25b: Honors Linear Algebra and Real Analysis II

April 30: Notes on line integrals and Green's Theorem

You may have wondered why we've suppressed the "dx" of single-variable integrals such as  $\int_a^b f(x) dx$  in the multivariate context. The reason is related to our relegating most of Chapter 5 to a course in differential geometry such as Math 136: "differentials" like  $dx$  and  $dy$  do make sense in higher dimension, but they interact in novel ways. Consider even what happens to an integral such as  $\iint f(x, y) dx dy$  under a linear change of variable with matrix  $\begin{pmatrix} a & b \\ a' & b' \end{pmatrix}$ : we should have  $d(ax + by) = a dx + b dy$ ,  $d(a'x + b'y) = a' dx + b' dy$ , and so we should have

$$\iint f(x, y) dx dy = \iint f(ax + by, a'x + b'y)(a dx + b dy)(a' dx + b' dy);$$

but we know that the actual formula has  $\pm(ab' - a'b) dx dy$ . Attempting to expand the product  $(a dx + b dy)(a' dx + b' dy)$  does yield a term  $ab' dx dy$ , but where  $-b'a dx dy$  should go we see instead  $ab' dy dx$ , and there are also the stray terms  $aa' dx dx$  and  $bb' dy dy$ . It turns out that "dx dy" is really " $dx \wedge dy$ ", where the  $\wedge$  (`\wedge`) is "exterior multiplication" which is *anticommutative*:  $dx \wedge dy = -dy \wedge dx$ , and  $dx \wedge dx = dy \wedge dy = 0$ . These rules do give the correct

$$(a dx + b dy) \wedge (a' dx + b' dy) = (ab' - a'b) dx \wedge dy$$

or minus that if the linear transformation  $\begin{pmatrix} a & b \\ a' & b' \end{pmatrix}$  is "orientation-reversing" (much as in single-variable calculus  $\int_s^r = -\int_r^s$ ). Exterior algebra, and more generally tensor algebra, is a beautiful part of abstract linear algebra, and even sheds new light on more familiar Math 25a topics such as determinants (you might check that indeed the anticommutativity does naturally yield

$$\begin{aligned} & (a_{11}dx + a_{12}dy + a_{13}dz) \wedge (a_{21}dx + a_{22}dy + a_{23}dz) \wedge (a_{31}dx + a_{32}dy + a_{33}dz) \\ &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} dx \wedge dy \wedge dz, \end{aligned}$$

and also the behavior of determinants under elementary row operations); but exterior algebra is not covered in Math 25a, and without it we cannot properly develop the theory of differential forms which is needed to even *state* Stokes' theorem. So for this last class meeting we content ourselves with an introduction that mostly uses just "differential 1-forms" and does not require us to wedge two differential forms together.

Stokes' theorem is a grand generalization of the Fundamental Theorem of Calculus (FTC), which states that if  $F$  is a  $C^1$  real-valued function on the interval  $[r, s]$  then  $F(s) - F(r) = \int_r^s F'(t) dt$ , a.k.a. the integral of "dF" from  $t = r$  to  $t = s$ . For our first attempt at a generalization, suppose  $U$  is an open subset of  $\mathbf{R}^n$  and  $\vec{r}, \vec{s} \in U$  are connected by a  $C^1$  path  $\gamma : [a, b] \rightarrow U$ , i.e.  $\gamma : [a, b] \rightarrow U$  is a  $C^1$  function such that  $\gamma(a) = \vec{r}$  and  $\gamma(b) = \vec{s}$ . [See the beginning of Edwards, Chapter V (and also the beginning of Chapter II) for more about paths in  $\mathbf{R}^n$ .] Then  $F \circ \gamma$  is a  $C^1$  real-valued function on  $[a, b]$ , so by FTC

$$F(\vec{s}) - F(\vec{r}) = (F \circ \gamma)(b) - (F \circ \gamma)(a) = \int_a^b (F \circ \gamma)'(t) dt,$$

and by the Chain Rule

$$(F \circ \gamma)'(t) dt = \nabla F(\gamma(t)) \cdot \gamma'(t) dt = \sum_{i=1}^n D_i F(\gamma(t)) \gamma'_i(t) dt.$$

This motivates our general definition: a *differential (1-)form* on  $U$  is an expression of the form

$$\omega = \sum_{i=1}^n f_i dx_i = f_1 dx_1 + f_2 dx_2 + \cdots + f_n dx_n,$$

where each  $f_i$  is a continuous real-valued function on  $U$ , and the *integral* of  $\omega$  on a  $\mathcal{C}^1$  path  $\gamma : [a, b] \rightarrow U$  is defined by

$$\int_{\gamma} \omega := \int_a^b \sum_{i=1}^n f_i(\gamma(t)) \gamma'_i(t) dt$$

[Cf. Edwards, page 295]. In particular if  $F : U \rightarrow \mathbf{R}$  is a  $\mathcal{C}^1$  function then the “differential of  $F$ ”, denoted  $dF$ , is the differential on  $U$  with each  $f_i$  being the corresponding partial derivative  $D_i F$ , and we’ve shown that  $\int_{\gamma} dF$  is the difference between the values of  $F$  at the endpoints of  $\gamma$ . [Edwards states and proves this as Theorem 1.5 on page 298.] NB once  $n > 1$  it is no longer the case that every differential is “exact”, i.e. of the form  $dF$  for some  $F$ ; if  $\omega$  is an exact differential then  $\int_{\gamma} \omega = 0$  for any *closed* path  $\gamma$  [a path  $\gamma : [a, b] \rightarrow \mathbf{R}^n$  with  $\gamma(b) = \gamma(a)$ ], and that certainly need not be the case for arbitrary  $f_i$  and  $\gamma$ . If  $\omega = \sum_i f_i dx_i$  is itself  $\mathcal{C}^1$  — that is, if each  $f_i$  is a  $\mathcal{C}^1$  function — then a necessary condition for  $\omega$  to be exact is that the partial derivatives of the  $f_i$  satisfy  $D_i f_j = D_j f_i$  for all  $i, j$ ; that’s because if  $\omega = dF$  then  $F$  is  $\mathcal{C}^2$ , whence its mixed partials commute:

$$D_i f_j = D_i D_j F = D_j D_i F = D_j f_i.$$

A  $\mathcal{C}^1$  differential satisfying this condition is said to be “closed” (sorry, this “closed” is not at all the same kind of thing as our familiar topological notion). We shall see that a closed differential form  $\omega$  is *locally* exact; but  $\omega$  might fail to be exact on all of  $U$ : Edwards (see Examples 3–4 on pages 296–298) gives the counterexample  $U = \{(x, y) \in \mathbf{R}^2 \mid (x, y) \neq (0, 0)\}$ ,  $\omega = (-y dx + x dy)/(x^2 + y^2)$  [that is,  $f_1 = -y/(x^2 + y^2)$  and  $f_2 = x/(x^2 + y^2)$ , which are the partial derivatives of the  $\theta$  of polar coordinates]: the integral of this  $\omega$  on the unit circle is  $2\pi \neq 0$ .

You might wonder at this point how we can “integrate  $\omega$  on the unit circle” without specifying  $\gamma$ . In fact one important feature of our definition of  $\int_{\gamma} \omega$  is that it is invariant under reparametrization of the path, as long as we preserve the same orientation along the path (in the case of the unit circle, this is the counterclockwise direction, which is usually the preferred “positive direction” in mathematics because it’s the direction of increasing  $\theta$ ). That is, if  $\tau$  is some increasing  $\mathcal{C}^1$  function  $[a', b'] \rightarrow [a, b]$  with  $a = \tau(a')$  and  $b = \tau(b')$  then  $\gamma \circ \tau : [a', b'] \rightarrow U$  is another  $\mathcal{C}^1$  path that goes through the same points of  $U$  and in the same direction, and then we can derive  $\int_{\gamma \circ \tau} \omega = \int_{\gamma} \omega$  by the Chain Rule and the change-of-variable formula for derivatives and integrals of functions of one variable. So for example if I chose to parametrize the unit circle as  $(\cos \theta, \sin \theta)$  for  $\theta \in [0, 2\pi]$ , rather than Edwards’  $(\cos 2\pi t, \sin 2\pi t)$  for  $t \in [0, 1]$ , the resulting integral would not change.

Our final example yields *Green’s theorem on a rectangle*. (See Edwards, “Lemma 2.1” on pages 311–312, for the special case that the rectangle is the unit square  $[0, 1] \times [0, 1]$ ; an arbitrary rectangle works in exactly the same way.) For some real numbers  $x_1, x_2, y_1, y_2$  with  $x_1 < x_2$  and  $y_1 < y_2$ , let  $B \subset \mathbf{R}^2$  be the box

$$[x_1, x_2] \times [y_1, y_2] = \{(x, y) \in \mathbf{R}^2 \mid x_1 \leq x \leq x_2, y_1 \leq y \leq y_2\};$$

and let  $\omega = f_1 dx + f_2 dy$  be a  $\mathcal{C}^1$  form on some open set  $U$  containing  $B$ . Consider the integral of  $\omega$  along the oriented boundary  $\partial B$  of  $B$ , which consists of the line segments from

$(x_1, y_1)$  to  $(x_1, y_2)$  to  $(x_2, y_2)$  to  $(x_2, y_1)$  and back to  $(x_1, y_2)$ . Parametrize each one naturally by its non-constant coordinate, i.e. by  $x, y, x, y$  respectively. The derivatives  $\gamma'$  are then respectively  $(1, 0), (0, 1), (-1, 0), (0, -1)$ . Therefore  $\int_{\partial B} \omega$  is

$$\begin{aligned} & \int_{x_1}^{x_2} f_1(x, y_1) dx + \int_{y_1}^{y_2} f_2(x_2, y) dy \\ & - \int_{x_1}^{x_2} f_1(x, y_2) dx - \int_{y_1}^{y_2} f_2(x_1, y) dy. \end{aligned}$$

We combine the integrals with respect to  $x$  to

$$\int_{x_1}^{x_2} (f_1(x, y_1) - f_1(x, y_2)) dx = \int_{x_1}^{x_2} \left( - \int_{y_1}^{y_2} D_2 f_1(x, y) dy \right) dx,$$

and combine the integrals with respect to  $y$  to

$$\int_{y_1}^{y_2} (f_2(x_2, y) - f_2(x_1, y)) dy = \int_{y_1}^{y_2} \left( + \int_{x_1}^{x_2} D_1 f_2(x, y) dx \right) dy,$$

in each case using FTC in the second step. By Fubini we thus have

$$\int_{\partial B} \omega = \iint_B (D_1 f_2 - D_2 f_1) dx dy = \iint_B d\omega.$$

Here “ $d\omega$ ” is just defined to be  $(D_1 f_2 - D_2 f_1) dx \wedge dy$ ; but note that this is consistent with our earlier remarks on exterior products: if  $dF = (D_1 F) dx + (D_2 F) dy$ , then it makes sense to declare

$$d\omega = d(f_1 dx + f_2 dy) = (D_1 f_1 dx + D_2 f_1 dy) \wedge dx + (D_1 f_2 dx + D_2 f_2 dy) \wedge dy$$

which yields  $(D_1 f_2 - D_2 f_1) dx \wedge dy$  using  $dx \wedge dx = dy \wedge dy = 0$  and  $dy \wedge dx = -dx \wedge dy$ .

Note that the condition that  $\omega$  be closed is precisely the condition  $d\omega = 0$ , which is satisfied automatically when  $\omega$  is exact (i.e. when  $\omega = dF$  for some  $C^2$  function  $F$ ). In that case we can deduce  $\int_{\partial B} \omega = 0$ . This turns out to generalize to higher differential forms as well:  $d$  maps  $k$ -forms to  $(k+1)$ -forms, and  $d \circ d$  is always zero. (A  $k$ -form is a sum of expressions  $f(\vec{x}) dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ , so an ordinary function is a 0-form while  $\omega$  is a 1-form and  $d\omega$  is a 2-form.)

See Edwards and the main M25b.13 webpage for some applications of Green’s theorem.