

## Math 25b: Honors Linear Algebra and Real Analysis II

### Topological proof of the Fundamental Theorem of Algebra

Let  $P \in \mathbf{C}[z]$  be a polynomial (with complex coefficients) of degree  $n > 0$ , say<sup>1</sup>

$$f(z) = \sum_{j=0}^n a_j z^j = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + a_n z^n$$

with  $a_n \neq 0$ . We have seen (in Math 25a) that once we prove that every such  $P$  has a single complex root (i.e., there exists a complex number  $r_1$  such that  $P(r_1) = 0$ ) it soon follows by induction that  $P$  factors completely:  $P(z) = a_n \prod_{m=1}^n (z - r_m)$  where the roots are  $r_1, r_2, \dots, r_n$ . But proving the existence of  $r_1$  is tricky. We use our topological tools to give a proof.

Start by observing that  $f$  is a continuous function from  $\mathbf{C}$  to  $\mathbf{C}$  (we can build this function out of the constant functions  $z \mapsto a_j$  and the identity function  $z \mapsto z$  by a finite sequence of additions and multiplications, each of which takes continuous functions to continuous functions). Therefore the real-valued function taking any  $z \in \mathbf{C}$  to  $|f(z)| = d(0, f(z))$  is continuous too. This function, which we shall call  $|f|$ , takes every complex number to a nonnegative real number, and if we can show that it attains the value zero at some  $z \in \mathbf{C}$  then we'll be done. We shall do this by first showing that  $|f|$  attains a minimum somewhere on  $\mathbf{C}$ , and then that if  $f$  is a polynomial then  $|f|$  cannot have a minimum value other than zero.

We shall prove that  $|f|$  attains its minimum using compactness. Note that this is not immediate from the continuity of  $|f|$  because  $\mathbf{C}$  is not compact! However, just as is the case for real-valued polynomials, once  $|z|$  is large so is  $|f(z)|$  so we need only consider  $|f|$  on a bounded subset of  $\mathbf{C}$ , and as long as we make that subset closed it will be compact (Heine-Borel) and we'll be able to proceed.

Here's how it works. Assume  $|z| \geq 1$ . By repeated application of the triangle inequality (and of the identity  $|wz| = |w||z|$ , i.e. the multiplicativity of the absolute value on  $\mathbf{C}$ ), we have

$$\begin{aligned} |f(z)| &= |a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + a_n z^n| \\ &\geq |a_n z^n| - |a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}|, \\ &= |a_n| |z|^n - |a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}|, \end{aligned}$$

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<sup>1</sup>NB it is a good idea to avoid using  $i$  as an index when working with complex numbers, which might lead to avoidable confusion with  $i = \sqrt{-1}$ .

and bound the second term  $\left| \sum_{j=0}^{n-1} a_j z^j \right|$  as follows:

$$\left| \sum_{j=0}^{n-1} a_j z^j \right| \leq \sum_{j=0}^{n-1} |a_j| |z|^j \leq \sum_{j=0}^{n-1} |a_j| |z|^{n-1} = \left( \sum_{j=0}^{n-1} |a_j| \right) \cdot |z|^{n-1}.$$

That is, we have a “main term”  $a_n z^n$  and an “error term” of size at most  $A|z|^{n-1}$ , where  $A = \sum_{j=0}^{n-1} |a_j|$ . So the absolute value  $|a_n| |z|^n$  of the “main term” exceeds that of the “error term” by a factor at least  $|a_n| |z|/A$ . By making  $|z|$  large enough we can make this factor as large as we want. Any ratio bigger than 1 will work; for example, if  $|z| > 2A/|a_n|$  then  $|f(z)| > \frac{1}{2}|a_n| |z|^n$ , and (since  $n > 0$ ) this gets arbitrarily large as  $|z|$  grows further.

In particular, we can find  $R$  large enough that  $R > 2A/|a_n|$  and also  $\frac{1}{2}|a_n|R^n$  exceeds a known value of  $|f|$ , such as  $|f(0)| = a_0$ . Then, once  $|z| \geq R$  we have

$$|f(z)| > \frac{1}{2}|a_n|R^n > a_0 = |f(0)|.$$

Thus, as promised, it is enough to restrict  $z$  to the closed disc  $|z| \leq R$ . This is a compact space, so  $|f|$  has a minimum value at some  $z_0$ , which is smaller than the value of  $|f(z)|$  not only for  $|z| \leq R$  but for all  $z \in \mathbf{C}$ : we need only check this if  $|z| > R$ , but for such  $z$  we have

$$|f(z)| > \frac{1}{2}|a_n|R^n > a_0 = |f(0)| \geq |f(z_0)|.$$

We next show that if  $|f(z)|$  is minimized at  $z = z_0$  then  $f(z_0) = 0$ . It is enough to do this for  $z_0 = 0$ , because in general if  $|f(z)|$  is minimized at  $z = z_0$  then  $|f(z + z_0)|$  is minimized at  $z = 0$ , and the function taking  $z$  to  $f(z + z_0)$  is a polynomial of the same degree as  $f$ .

We shall show that if  $f(0) \neq 0$  then  $|f|$  cannot even have a “local minimum” at  $z = 0$ ; i.e., for every  $\epsilon > 0$  there exists  $z$  such that  $|z| < \epsilon$  and  $|f(z)| < |f(0)|$ . The argument begins similarly to our analysis of  $|f(z)|$  for large  $|z|$ . Again we write  $f(z) = \sum_{j=0}^n a_j z^j$ , and this time ask what happens for small  $|z|$ . By hypothesis the constant term  $a_0 = f(0)$  is nonzero. Also  $a_n \neq 0$ , so there is some  $k > 0$  such that  $a_k \neq 0$  but  $a_1 = a_2 = \cdots = a_{k-1} = 0$ .<sup>2</sup> We claim that for small  $|z|$  we can approximate  $f(z)$  by  $a_0 + a_k z^k$  to within an error smaller than half of  $|a_k z^k|$ . Indeed the error has size

$$|a_{k+1}z^{k+1} + a_{k+2}z^{k+2} + \cdots + a_n z^n| \leq |a_{k+1}z^{k+1}| + |a_{k+2}z^{k+2}| + \cdots + |a_n z^n|,$$

and each term  $|a_j z^j|$  with  $j \geq k+1$  is  $|a_j| |z|^j$ , which is at most  $|a_j| |z|^{k+1}$  if  $|z| \leq 1$ , so the sum is at most  $B|z|^{k+1}$  where  $B = \sum_{j=k+1}^n |a_j|$ . Thus for  $|z| < r$  we have

$$|f(z) - (a_0 + a_k z^k)| \leq \frac{1}{2}|a_k z^k|$$

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<sup>2</sup>Try  $k = 1$ ; if that doesn't work then  $a_1 = 0$ , so try  $k = 2$ ; if that doesn't work then  $a_1 = a_2 = 0$ , so try  $k = 3$ , etc.; eventual success is guaranteed by  $a_n \neq 0$ .

provided that  $r$  is small enough that  $r < 1$  and  $Br \leq \frac{1}{2}|a_k|$ . (Note that since  $a_k \neq 0$  there are positive  $r$  satisfying this condition.)

So far everything we've done would work just as well for real polynomials, which don't have to have real zeros (for example,  $x^2 + 1$  is minimized in absolute value at  $x = 0$  where it equals  $1 \neq 0$ ). We now use the following feature of  $\mathbf{C}$ : as  $z$  varies over the circle of radius  $r$ , the value of  $a_k z^k$  varies over the circle of radius  $|a_k|r^k$  and attains every value on that circle, including the value that makes  $|a_0 + a_k z^k| = |a_0 - |a_k|r^k|$ . (In the real numbers this fails if  $k$  is even!) Indeed we can use "de Moivre's formula"

$$r(\cos \theta + i \sin \theta)^k = r(\cos k\theta + i \sin k\theta)$$

to see that every point on the circle of radius  $|a_k|r^k$  arises for  $k$  choices of  $z$ .

For  $r$  small enough,  $|a_k|r^k < |a_0|$ , so  $|a_0 + a_k z^k| = |a_0| - |a_k|r^k$ . As long as  $r$  is small enough, we conclude that

$$|f(z)| \leq |a_0 + a_k z^k| + \frac{1}{2}|a_k z^k| = |a_0| - |a_k|r^k + \frac{1}{2}|a_k|r^k = |a_0| - \frac{1}{2}|a_k|r^k < |a_0| = |f(0)|,$$

and we've found a value of  $|f|$  that's smaller than the purported minimum at  $z = 0$ . We have thus reached a contradiction and completed the proof of the Fundamental Theorem of Algebra.