Math 259: Introduction to Analytic Number Theory

Exponential sums III: the van der Corput inequalities

Let $f(x)$ be a sufficiently differentiable function, and $S = \sum_{n=1}^{N} e(f(n))$. The Kuzmin inequality tells us in effect that

I If $f'(x)$ is monotonic and continuous with $\|f'(x)\| > \lambda_1$ for $x \in [1, N]$ then $S \ll 1/\lambda_1$.

(Recall that $\|x\|$ is the distance from the real number x to the nearest integer.) We shall use this inequality to deduce van der Corput's estimates on S in terms of N and higher derivatives of f. In each case the inequality is useful only if f has a derivative $f^{(k)}$ of constant sign which is significantly smaller than 1.

II If there are constants c, C with $0 < c < C$ such that $c\lambda_2 < f'' < C\lambda_2$ for all $x \in [1, N]$ then

$$S \ll_{c,C} N\lambda_2^{1/2} + \lambda_2^{-1/2}.$$

III If there are constants c, C with $0 < c < C$ such that $c\lambda_3 < f''' < C\lambda_3$ for all $x \in [1, N]$ then

$$S \ll_{c,C} N\lambda_3^{1/6} + N^{1/2}\lambda_3^{-1/6}.$$

In general there is a k-th inequality

$$S \ll_{c,C} N\lambda_k^{1/(2^k-2)} + N^{1-2^{2-k}}\lambda_k^{-1/(2^k-2)}$$

when $c\lambda_k < f^{(k)} < C\lambda_k$ for all $x \in [1, N]$, but we'll make use only of van der Corput **II** and **III**.

Here is a typical application, due to van der Corput.

Theorem. *We have* $\zeta(1/2 + it) \ll |t|^{1/6} \log |t|$ *for all* $t \in \mathbf{R}$ *such that* $|t| > 2$.

Proof (assuming **II** and **III**): We have seen that

$$\zeta(1/2 + it) = \sum_{n=1}^{\lfloor |t|/\pi \rfloor} n^{-1/2-it} + O(1).$$

We break up the sum into segments $\sum_{n=N}^{N_1}$ with $N < N_1 \le 2N$, and use $f(x) = (t \log x)/2\pi$, so on each segment $\lambda_k = t/N^k$ holds for either f or $-f$. Then **II** and **III** give

$$\sum_{n=N}^{N'} n^{it} \ll |t|^{1/2} + N/|t|^{1/2}, \qquad \sum_{n=N}^{N'} n^{it} \ll N^{1/2}|t|^{1/6} + N/|t|^{1/6}$$

for $N < N' < N_1$. By partial summation, it follows that

$$\sum_{n=N}^{N'} n^{-1/2-it} \ll (|t|/N)^{1/2} + (N/|t|)^{1/2}, \qquad \sum_{n=N}^{N'} n^{-1/2-it} \ll |t|^{1/6} + N^{1/2}/|t|^{1/6}$$

Choosing the first estimate for $N \gg |t|^{2/3}$ and the second for $N \ll |t|^{2/3}$ we find that the sum is $\ll |t|^{1/6}$ in either case. Since the total number of $[N, N']$ segments is $O(\log |t|)$, the inequality $\zeta(1/2 + it) \ll |t|^{1/6} \log |t|$ follows.

The inequality **II** is an easy consequence of Kuzmin's **I**. [NB the following is not van der Corput's original proof, for which see for instance Lecture 3 of [Montgomery 1994]. The proof we give is much more elementary, but does not as readily yield the small further reductions of the exponents that are available with the original method.] We may assume that $f''(x) < 1/4$ on $[1, N]$, else $\lambda_2 \gg 1$ and the inequality is trivial. Split $[1, N]$ into $O(N\lambda_2 + 1)$ intervals on which $\lfloor f' \rfloor$ is constant. Let λ_1 be a small positive number to be determined later, and take out $O(N\lambda_2 + 1)$ subintervals of length $O((\lambda_1/\lambda_2) + 1)$ on which f' is within λ_1 of an integer. On each excised interval, estimate the sum trivially by its length; on the remaining intervals, use Kuzmin. This yields

$$S \ll (N\lambda_2 + 1)(\lambda_1^{-1} + (\lambda_1/\lambda_2) + 1).$$

Now take $\lambda_1 = \lambda_2^{1/2}$ to get

$$S \ll (N\lambda_2 + 1)(\lambda_2^{-1/2} + 1).$$

But by assumption $\lambda_2 \ll 1$, so the second factor is $\ll \lambda_2^{-1/2}$. This completes the proof of **II**.

For **III** and higher van der Corput bounds, we shall follow Weyl by showing that

$$S \ll \left\{ \frac{N}{H} \sum_{h=0}^{H} \left| \sum_{n=1}^{N-h} e(f(n+h) - f(n)) \right| \right\}^{1/2}. \tag{1}$$

for $H \leq N$. If $f(x)$ has small positive k-th derivative then each $f(x+h) - f(x)$ has small $(k-1)$-st derivative, which is positive except for $h = 0$ when the inner sum is N. This will let us prove **III** from **II**, and so on by induction (see the first Exercise below).

To prove (1), define z_n for $n \in \mathbf{Z}$ by $z_n = e(f(n))$ for $1 \leq n \leq N$ and $z_n = 0$ otherwise. Then

$$S = \sum_{n=-\infty}^{\infty} z_n = \frac{1}{H} \sum_{n=-\infty}^{\infty} \left(\sum_{h=1}^{H} z_{n+h} \right),$$

in which fewer than $N + H$ of the inner sums are nonzero. Thus by the (Cauchy-)Schwarz inequality,

$$|S|^2 \leq \frac{N+H}{H^2} \sum_{n=-\infty}^{\infty} \left| \sum_{h=1}^{H} z_{n+h} \right|^2 \ll \frac{N}{H^2} \sum_{h_1, h_2 = 1}^{H} \left| \sum_{n \in \mathbf{Z}} z_{n+h_1} \overline{z_{n+h_2}} \right|.$$

But the inner sum depends only on $|h_1 - h_2|$, and each possible $h := h_1 - h_2$ occurs at most H times. So,

$$|S|^2 \ll \frac{N}{H} \sum_{h=0}^{H} \left| \sum_{n \in \mathbf{Z}} z_{n+h} \overline{z_n} \right|,$$

from which (1) follows.

Now to prove **III**: we may assume $N^{-3} < \lambda_3 < 1$, else the inequality is trivial. Apply (1), and to each of the inner sums with $h \neq 0$ apply **II** with $\lambda_2 = h\lambda_3$. This yields

$$\begin{aligned} |S|^2 &\ll \frac{N^2}{H} + \frac{N}{H} \sum_{h=1}^{H} [N(h\lambda_3)^{1/2} + (h\lambda_3)^{-1/2}] \\ &= N^2\big((H\lambda_3)^{1/2} + H^{-1}\big) + N/(H\lambda_3)^{1/2}. \end{aligned}$$

Now make the first two terms equal by taking $H = \lfloor \lambda_3^{-1/3} \rfloor$:

$$|S|^2 \ll N^2 \lambda_3^{1/3} + N \lambda_3^{-1/3}.$$

Extracting square roots yields **III**.

Exercises

1. Prove the van der Corput estimates **IV**, **V**, etc. by induction.

2. Prove that $\{\log_b n!\}_{n=0}^{\infty}$ is equidistributed mod 1 for any $b > 1$.

3. Use (1) to prove the equidistribution of $\{nP(n)\}$ mod 1 for any polynomial $P(x)$ with an irrational coefficient (which was Weyl's original application of (1)). Give necessary and sufficient conditions on polynomials $P_1, P_2, \ldots, P_k \in \mathbf{R}[x]$ for the sequence of vectors $(P_1(n), P_2(n), \ldots, P_k(n))$ to be equidistributed mod $\mathbf{Z}^k$.

Reference

[Montgomery 1994] Montgomery, H.L.: *Ten lectures on the interface between analytic number theory and harmonic analysis.* Providence: AMS, 1994 [AB 9.94.9].