Math 259: Introduction to Analytic Number Theory

pseudo-syllabus

0. Introduction: What is analytic number theory?

1. Distribution of primes before complex analysis: classical techniques (Euclid, Euler); primes in arithmetic progressions via Dirichlet characters and L-series; Čebyšev's estimates on $\pi(x)$.

2. Distribution of primes using complex analysis: $\zeta(s)$ and $L(s,\chi)$ as functions of a complex variable, and the proof of the Prime Number Theorem and its extension to Dirichlet; blurb for Čebotarev density; functional equations; the Riemann hypothesis, extensions, generalizations and consequences.

3. Selberg's quadratic sieve and applications.

4. Analytic estimates on exponential sums (van der Corput etc.); prototypical applications: Weyl equidistribution, upper bounds on $|\zeta(s)|$ and $|L(s,\chi)|$ on vertical lines, lattice point sums.

5. Lower bounds on discriminants, conductors, etc. from functional equations; geometric analogue: how many points can a curve of genus $g \to \infty$ have over a given finite field?

6. Analytic bounds on coefficients of modular forms and functions; applications to counting representations of integers as sums of squares, etc.

Prerequisites While Math 259 will proceed at a pace appropriate for a graduate-level course, its prerequisites are perhaps surprisingly few: complex analysis at the level of Math 113, and linear algebra and basic number theory (up to say arithmetic in the field $\mathbf{Z}/p\mathbf{Z}$ and Quadratic Reciprocity). Some considerably deeper results (such as estimates on Kloosterman sums) will be cited but may be regarded as black boxes for our purposes. If you know about algebraic number fields or modular forms or curves over finite fields, you'll get more from the course at specific points, but these points will be in the nature of scenic detours that are not required for the main journey.

Texts Lecture notes will be handed out periodically, and can also be found on the course webpage. There is no textbook: this class is an introduction to several different flavors of analytic methods in number theory, and I know of no one work that covers all this material. Thus I intend to expand and edit the lecture notes to put together a textbook, which may become available by the next time I teach the class… Supplementary readings such as Serre's *A Course in Arithmetic* and Titchmarsh's *The Theory of the Riemann Zeta-Function* will be suggested as we approach their respective territories.

Office Hours 335 Sci Ctr, Thursdays 2:45–4:15 PM (occasionally shortened by Colloquium or faculty meetings), or e-mail me at `elkies@math` (`elkies@math.harvard.edu` from outside Harvard) to ask questions or set up an alternative meeting time.

Grading Assuming there are enough students taking the class for a grade (i.e., who are not a post-Qual math graduate student exercising their EXC option), there *will* be weekly homework (unlike in previous editions of this class), drawn from exercises in the lecture notes. This will account for about 2/3 of your grade; as usual in our department, you are allowed — indeed encouraged — to collaborate on solving homework problems, but must write up your own solutions. The remaining 1/3 of the grade will come from a final project, which (again depending on class size and composition) will be an expository paper and/or an in-class presentation on some aspect of analytic number theory related to but just beyond what we cover in class. I intend to put together some Extended Exercises that can be used as final topics; the supplementary references will be another good source for paper/presentation topics.

Misc. I'll have to miss several class meetings, notably the entire week of February 12–16. I'll make up for most of the lost time during Reading Period.