

## Math 259: Introduction to Analytic Number Theory

More about the Gamma function

We collect some more facts about  $\Gamma(s)$  as a function of a complex variable that will figure in our treatment of  $\zeta(s)$  and  $L(s, \chi)$ . All of these, and most of the Exercises, are standard textbook fare; one basic reference is Ch. XII (pp. 235–264) of [WW 1940]. One reason for not just citing Whittaker & Watson is that some of the results concerning Euler's integrals  $B$  and  $\Gamma$  have close analogues in the Gauss and Jacobi sums associated to Dirichlet characters, and we shall need these analogues before long.

**The product formula for  $\Gamma(s)$ .** Recall that  $\Gamma(s)$  has simple poles at  $s = 0, -1, -2, \dots$  and no zeros. We readily concoct a product that has the same behavior: let

$$g(s) := \frac{1}{s} \prod_{k=1}^{\infty} e^{s/k} / \left(1 + \frac{s}{k}\right),$$

the product converging uniformly in compact subsets of  $\mathbf{C} - \{0, -1, -2, \dots\}$  because  $e^x/(1+x) = 1 + O(x^2)$  for small  $x$ . Then  $\Gamma/g$  is an entire function with neither poles nor zeros, so it can be written as  $\exp \alpha(s)$  for some entire function  $\alpha$ . We show that  $\alpha(s) = -\gamma s$ , where  $\gamma = 0.57721566490\dots$  is *Euler's constant*:

$$\gamma := \lim_{N \rightarrow \infty} \left( -\log N + \sum_{k=1}^N \frac{1}{k} \right).$$

That is, we show:

**Lemma.** *The Gamma function has the product formulas*

$$\Gamma(s) = e^{-\gamma s} g(s) = \frac{e^{-\gamma s}}{s} \prod_{k=1}^{\infty} e^{s/k} / \left(1 + \frac{s}{k}\right) = \frac{1}{s} \lim_{N \rightarrow \infty} \left( N^s \prod_{k=1}^N \frac{k}{s+k} \right). \quad (1)$$

*Proof:* For  $s \neq 0, -1, -2, \dots$ , the quotient  $g(s+1)/g(s)$  is the limit as  $N \rightarrow \infty$  of

$$\begin{aligned} \frac{s}{s+1} \prod_{k=1}^N e^{1/k} \frac{1 + \frac{s}{k}}{1 + \frac{s+1}{k}} &= \frac{s}{s+1} \left( \exp \sum_{k=1}^N \frac{1}{k} \right) \prod_{k=1}^N \frac{k+s}{k+s+1} \\ &= s \cdot \frac{N}{N+s+1} \cdot \exp \left( -\log N + \sum_{k=1}^N \frac{1}{k} \right). \end{aligned}$$

Now the factor  $N/(N+s+1)$  approaches 1, while  $-\log N + \sum_{k=1}^N \frac{1}{k} \rightarrow \gamma$ . Thus  $g(s+1) = s e^{\gamma} g(s)$ , and if we define  $\Gamma^?(s) := e^{-\gamma s} g(s)$  then  $\Gamma^?$  satisfies the same functional equation  $\Gamma^?(s+1) = s \Gamma^?(s)$  satisfied by  $\Gamma$ . We are claiming that in fact  $\Gamma^? = \Gamma$ .

Consider  $q := \Gamma/\Gamma^?$ , an entire function of period 1. Thus it is an analytic function of  $e^{2\pi is} \in \mathbf{C}^*$ . We wish to show that  $q = 1$  identically. By the definition of  $g$  we have  $\lim_{s \rightarrow 0} sg(s) = 1$ ; hence

$$\lim_{s \rightarrow 0} s\Gamma^?(s) = \lim_{s \rightarrow 0} sg(s) = 1 = \lim_{s \rightarrow 0} s\Gamma(s),$$

and  $q(0) = 1$ . We claim that there exists a constant  $C$  such that

$$|q(\sigma + it)| \leq Ce^{\pi|t|/2} \quad (2)$$

for all real  $\sigma, t$ ; since the coefficient  $\pi/2$  in the exponent is less than  $2\pi$ , it will follow that  $q$  is constant, and thus that  $\Gamma^? = \Gamma$  as claimed.

Since  $q$  is periodic, we need only prove (2) for  $s = \sigma + it$  with  $\sigma \in [1, 2]$ . For such  $s$ , we have  $|\Gamma(\sigma + it)| \leq \Gamma(\sigma)$  by the integral formula and

$$\left| \frac{\Gamma^?(\sigma + it)}{\Gamma^?(\sigma)} \right| = \prod_{k=0}^{\infty} \frac{\sigma + k}{|\sigma + k + it|} = \exp \left( -\frac{1}{2} \sum_{k=0}^{\infty} \log \left( 1 + \frac{t^2}{(\sigma + k)^2} \right) \right).$$

The summand is a decreasing function of  $k$ , so the sum is

$$\leq \int_0^{\infty} \log(1 + (t/x)^2) dx = |t| \int_0^{\infty} \log(1 + (1/x)^2) dx,$$

which on integration by parts becomes  $2|t| \int_0^{\infty} dx/(x^2 + 1) = \pi|t|$ . This proves (2) with  $C = \sup_{1 \leq \sigma \leq 2} q(\sigma)$ , and completes the proof of (1).  $\square$

**Consequences of the product formula.** Our most important application of the product formula for  $\Gamma(s)$  is the *Stirling approximation*<sup>1</sup> to  $\log \Gamma(s)$ . Fix  $\epsilon > 0$  and let  $R_{\epsilon}$  be the region

$$\{s \in \mathbf{C}^* : |\operatorname{Im}(\log s)| < \pi - \epsilon\}.$$

Then  $R_{\epsilon}$  is a simply-connected region containing none of the poles of  $\Gamma$ , so there is an analytic function  $\log \Gamma$  on  $R_{\epsilon}$ , real on  $R_{\epsilon} \cap \mathbf{R}$ , and given by the above product formula:

$$\log \Gamma(s) = \lim_{N \rightarrow \infty} \left( s \log N + \log N! - \sum_{k=0}^N \log(s + k) \right). \quad (3)$$

We prove:

**Lemma.** *The approximation*

$$\log \Gamma(s) = \left(s - \frac{1}{2}\right) \log s - s + \frac{1}{2} \log(2\pi) + O_{\epsilon}(|s|^{-1}) \quad (4)$$

holds for all  $s$  in  $R_{\epsilon}$ .

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<sup>1</sup>Originally only for  $n! = \Gamma(n+1)$ , but we need it for complex  $s$  as well.

*Proof:* The estimate holds for small  $s$ , say  $|s| < 1$ , because  $O_\epsilon(|s|^{-1})$  well exceeds all the other terms. We thus assume  $|s| \geq 1$ , and estimate the sum in (3) as we did for  $\log x!$  in obtaining the original form of Stirling's approximation. The sum differs from

$$\begin{aligned} \int_{-\frac{1}{2}}^{N+\frac{1}{2}} \log(s+x) dx &= (N + \frac{1}{2} + s) \log(N + \frac{1}{2} + s) - (s - \frac{1}{2}) \log(s - \frac{1}{2}) - N - 1 \\ &= (N + \frac{1}{2} + s) \log N + (N + \frac{1}{2} + s) \log(1 + \frac{1}{N}(s + \frac{1}{2})) - (s - \frac{1}{2}) \log(s - \frac{1}{2}) - N - 1 \end{aligned}$$

by

$$\frac{1}{2} \int_{-\frac{1}{2}}^{N+\frac{1}{2}} \frac{\|x + \frac{1}{2}\|^2}{(s+x)^2} dx \ll_\epsilon |s|^{-1}.$$

We already know that  $\log N! = (N + 1/2) \log N - N + A + O(N^{-1})$  for some constant  $A$ . The estimate (4) follows upon taking  $N \rightarrow \infty$ , except for the value  $\frac{1}{2} \log(2\pi)$  of the constant term. This constant can be obtained by letting  $s \rightarrow \infty$  in the duplication formula  $\Gamma(2s) = \pi^{-1/2} 2^{2s-1} \Gamma(s) \Gamma(s + \frac{1}{2})$ .  $\square$

One can go on to expand the  $O_\epsilon(|s|^{-1})$  error in an asymptotic series in inverse powers of  $s$  (see the Exercises), but (4) is already more than sufficient for our purposes, in that we do not need the identification of the constant term with  $\frac{1}{2} \log 2\pi$ .

The logarithmic derivative of our product formula for  $\Gamma(s)$  is

$$\frac{\Gamma'(s)}{\Gamma(s)} = -\gamma - \frac{1}{s} + \sum_{k=1}^{\infty} \left( \frac{1}{k} - \frac{1}{s+k} \right) = \lim_{N \rightarrow \infty} \left[ \log N - \sum_{k=0}^N \frac{1}{s+k} \right].$$

Either by differentiating<sup>2</sup> (4) or by applying the same Euler-Maclaurin step to  $\sum_0^N 1/(s+k)$  we find that

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s - \frac{1}{2s} + O_\epsilon(|s|^{-2}). \quad (5)$$

### Remark

The product formula for  $\Gamma(s)$  can also be obtained for real  $s$  by elementary means, starting from the characterization of  $\Gamma$  as the unique logarithmically convex function on  $(0, \infty)$  satisfying the recursion  $\Gamma(s+1) = s\Gamma(s)$  and normalized by  $\Gamma(1) = 1$  (the Bohr-Mollerup theorem, see for instance [Rudin 1976, p.193]). The theorem for complex  $s$  can then be obtained by analytic continuation. The method used here, though less elegant, generalizes to a construction

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<sup>2</sup>While real asymptotic series cannot in general be differentiated (why?), complex ones can, thanks to Cauchy's integral formula for the derivative. The logarithmic derivative of  $\Gamma(s)$  is often called  $\psi(s)$  in the literature, but alas we cannot use this notation because it conflicts with  $\psi(x) = \sum_{n < x} \Lambda(n) \dots$

of product formulas for a much more general class of functions, as we shall see next.

### Exercises

On the product formula:

1. Verify that the duplication formula for  $\Gamma(2s)$  yields the correct constant term in (4). Apply Euler-Maclaurin to the sum in (3) to show that the  $O_\epsilon(|s|^{-1})$  error can be expanded in an asymptotic series in inverse powers of  $s$ .
2. Use (1) to obtain a product formula for  $\Gamma(s)\Gamma(-s)$ , and deduce that

$$\Gamma(s)\Gamma(1-s) = \pi / \sin \pi s. \quad (6)$$

(This can also be obtained from  $\Gamma(s)\Gamma(1-s) = B(s, 1-s)$  by using the change of variable  $x = y/(y-1)$  in the Beta integral and evaluating the resulting expression by contour integration.) Use this together with the duplication formula and Riemann's formula for  $\zeta(1-s)$  to obtain the equivalent asymmetrical form

$$\zeta(1-s) = \pi^{-s} 2^{1-s} \Gamma(s) \cos \frac{\pi s}{2} \zeta(s)$$

of the functional equation for  $\zeta(s)$ . Note that the duplication formula, and its generalization

$$\Gamma(ns) = (2\pi)^{\frac{1-n}{2}} n^{ns-\frac{1}{2}} \prod_{k=0}^{n-1} \Gamma\left(s + \frac{k}{n}\right),$$

can also be obtained from (1).

3. Show that  $\log \Gamma(s)$  has the Taylor expansion

$$\log \Gamma(s) = -\gamma(s-1) + \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n)(s-1)^n$$

about  $s = 1$ . Recover from this the Laurent expansion

$$\Gamma(s) = \frac{1}{s} - \gamma + \left(\gamma^2 + \frac{\pi^2}{6}\right) \frac{s}{2} + O(s^2)$$

of  $\Gamma(s)$  about  $s = 0$ .

Behavior of  $\Gamma(s)$  on vertical lines:

4. Deduce from (4) that for fixed  $\sigma \in \mathbf{R}$

$$\operatorname{Re}(\log \Gamma(\sigma + it)) = \left(\sigma - \frac{1}{2}\right) \log |t| - \frac{\pi}{2} |t| + C_\sigma + O_\sigma(|t|^{-1})$$

as  $|t| \rightarrow \infty$ . Check that for  $\sigma = 0, 1/2$  this agrees with the exact formulas

$$|\Gamma(it)|^2 = \frac{\pi}{t \sinh \pi t}, \quad |\Gamma(1/2 + it)|^2 = \frac{\pi}{\cosh \pi t}$$

obtained from (6).

5. For  $a, b, c > 0$ , determine the Fourier transform of  $f(x) = \exp(ax - be^{cx})$ , and check your answer by using contour integration to calculate the Fourier transform of  $\hat{f}$ . Now apply Poisson summation, let  $a \rightarrow 0$  and  $C = e^c > 1$ , and describe the behavior of  $\sum_{n=0}^{\infty} z^{C^n}$  as  $z \rightarrow 1$  from below. What does

$$\sum_{n=0}^{\infty} (-1)^n z^{2^n} = z - z^2 + z^4 - z^8 + z^{16} - + \dots$$

do as  $z \rightarrow 1$ ? Use this to prove that  $\mathbf{Z} \cap \bigcup_{m=0}^{\infty} [2^{2^m}, 2^{2^{m+1}})$  is an explicit example of a set of integers that does not have a logarithmic density.

An alternative proof of the functional equation for  $\zeta(s)$  (also in Riemann's fundamental paper of 1859):

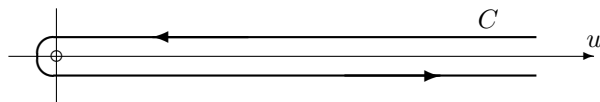
6. Prove that

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} u^{s-1} \frac{du}{e^u - 1}$$

for  $\sigma > 1$ , and that when  $s$  is not a positive integer an equivalent formula is

$$\zeta(s) = -\frac{e^{-\pi i s}}{2\pi i} \Gamma(1-s) \int_C u^{s-1} \frac{du}{e^u - 1}$$

where  $C$  is a contour coming from  $+\infty$ , going counterclockwise around  $u = 0$ , and returning to  $+\infty$ :



Show that this gives the analytic continuation of  $\zeta$  to a meromorphic function on  $\mathbf{C}$ ; shift the line of integration to the left to obtain the functional equation relating  $\zeta(s)$  to  $\zeta(1-s)$  for  $\sigma < 0$ , and thus for all  $s$  by analytic continuation.

## References

[WW 1940] Whittaker, E.T., Watson, G.N.: *A Course of Modern Analysis*.<sup>3</sup> (fourth edition). Cambridge University Press, 1940 (reprinted 1963). [HA 9.40 / QA295.W38]

[Rudin 1976] Rudin, W.: *Principles of Mathematical Analysis* (3rd edition). New York: McGraw-Hill, 1976.

<sup>3</sup>The full title is 26 words long, which was not out of line when the book first appeared in 1902. You can find the title in Hollis.