

Math 259: Introduction to Analytic Number Theory

A nearly zero-free region for $L(s, \chi)$, and Siegel's theorem

We used positivity of the logarithmic derivative of ζ_q to get a crude zero-free region for $L(s, \chi)$. Better zero-free regions can be obtained with some more effort by working with the $L(s, \chi)$ individually. The situation is most satisfactory for complex χ , that is, for characters with $\chi^2 \neq \chi_0$. (Recall that real χ were also the characters that gave us the most difficulty in the proof of $L(1, \chi) \neq 0$; it is again in the neighborhood of $s = 1$ that it is hard to find a good zero-free region for the L -function of a real character.)

To obtain the zero-free region for $\zeta(s)$, we started with the expansion of the logarithmic derivative

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \Lambda(n)n^{-s} \quad (\sigma > 1)$$

and applied the inequality

$$0 \leq \frac{1}{2}(z + 2 + \bar{z})^2 = \operatorname{Re}(z^2 + 4z + 3) = 3 + 4\cos\theta + \cos 2\theta \quad (z = e^{i\theta})$$

to the phases $z = n^{-it} = e^{i\theta}$ of the terms n^{-s} . To apply the same inequality to

$$-\frac{L'}{L}(s, \chi) = \sum_{n=1}^{\infty} \chi(n)\Lambda(n)n^{-s},$$

we must use $z = \chi(n)n^{-it}$ instead of n^{-it} , obtaining

$$0 \leq \operatorname{Re}(3 + 4\chi(n)n^{-it} + \chi^2(n)n^{-2it}).$$

Multiplying by $\Lambda(n)n^{-\sigma}$ and summing over n yields

$$0 \leq 3 \left[-\frac{L'}{L}(\sigma, \chi_0) \right] + 4 \operatorname{Re} \left[-\frac{L'}{L}(\sigma + it, \chi) \right] + \operatorname{Re} \left[-\frac{L'}{L}(\sigma + 2it, \chi^2) \right]. \quad (1)$$

Now we see why the case $\chi^2 = \chi_0$ will give us trouble near $s = 1$: for such χ the last term in (1) is within $O(1)$ of $\operatorname{Re}(-(\zeta'/\zeta)(\sigma + 2it))$, so the pole of $(\zeta'/\zeta)(s)$ at $s = 1$ will undo us for small $|t|$.

Let us see how far (1) does take us. Our bounds will involve $\log q$ for small $|t|$, and $\log q|t|$ for large $|t|$. To cover both ranges, and also to accommodate the case $q = 1$ (the Riemann zeta function), we use the convenient and conventional abbreviation

$$\mathcal{L} := \log q(|t| + 2).$$

We shall prove:

Theorem. *There is a constant $c > 0$ such that if $L(\sigma + it, \chi) = 0$ for some primitive complex Dirichlet character $\chi \bmod q$ then*

$$\sigma < 1 - \frac{c}{\mathcal{L}}. \quad (2)$$

If χ is a real primitive character then (2) holds for all zeros of $L(s, \chi)$ with at most one exception. The exceptional zero, if it exists, is real and simple.

Proof: We again apply (1) for suitable σ, t with $1 < \sigma \leq 2$. The first term is

$$\leq 3 \left[-\frac{\zeta'}{\zeta}(\sigma) \right] < \frac{3}{\sigma - 1} + O(1).$$

For the remaining terms, we use the partial-fraction expansion

$$-\frac{L'}{L}(s, \chi) = \frac{1}{2} \log \frac{q}{\pi} + \frac{1}{2} \frac{\Gamma'}{\Gamma}((s + \mathfrak{a})/2) - B_\chi - \sum_{\rho} \left(\frac{1}{s - \rho} + \frac{1}{\rho} \right).$$

To eliminate the contributions of B_χ and $\sum_{\rho} 1/\rho$ we use this formula to evaluate $(L'/L)(2, \chi) - (L'/L)(s, \chi)$. By the Euler product we have $(L'/L)(2, \chi) = O(1)$. This leaves

$$-\frac{L'}{L}(s, \chi) = \frac{1}{2} \frac{\Gamma'}{\Gamma}((s + \mathfrak{a})/2) - \sum_{\rho} \left(\frac{1}{s - \rho} - \frac{1}{2 - \rho} \right) + O(1).$$

Next take real parts. For ρ of real part in $[0, 1]$ we have $\operatorname{Re}(1/(2 - \rho)) \ll |2 - \rho|^{-2}$. To estimate the sum of this over all ρ , we may apply Jensen's theorem to $\xi(2 + s, \chi)$, finding that the number (with multiplicity) of $|\rho|$ at distance at most r from 2 is $O(r \log qr)$, and thus by partial summation that $\sum_{\rho} |2 - \rho|^{-2} \ll \log q$. We estimate the real part of the Γ'/Γ term by Stirling as usual, and find

$$\operatorname{Re} \left[-\frac{L'}{L}(s, \chi) \right] < O(\mathcal{L}) - \sum_{\rho} \operatorname{Re} \frac{1}{s - \rho}.$$

Again each of the $\operatorname{Re}(1/(s - \rho))$ is nonnegative, so the estimate remains true if we include only some of the zeros ρ , or none of them.

In particular it follows that

$$\operatorname{Re} \left[-\frac{L'}{L}(\sigma + 2it, \chi^2) \right] < O(\mathcal{L}), \quad (3)$$

at least when χ^2 is a primitive character. If χ^2 is not primitive, but still not the trivial character χ_0 , then (3) holds when χ^2 is replaced by its corresponding primitive character; but the error thus introduced is at most

$$\sum_{p|q} \frac{p^{-\sigma}}{1 - p^{-\sigma}} \log p < \sum_{p|q} \log p \leq \log q < \mathcal{L},$$

so can be absorbed into the $O(\mathcal{L})$ error. But when $\chi^2 = \chi_0$ the partial-fraction expansion of its $-L'/L$ has a term $+1/(s-1)$ which cannot be discarded, and can be absorbed into $O(\mathcal{L})$ only if s is far enough from 1. We thus conclude that (3) holds unless $\chi^2 = \chi_0$ and $|t| < c/\log q$, the implied constant in (3) depending on c . (Equivalently, we could change $O(\mathcal{L})$ to $1/|t| + O(\mathcal{L})$ when $\chi^2 = \chi_0$.)

The endgame is the same as we have seen for the classical zero-free region for $\zeta(s)$: if there is a zero $\rho = 1 - \delta + it$ with δ small, use its imaginary part t in (1) and find from the partial-fraction expansion that

$$\operatorname{Re} \left[-\frac{L'}{L}(\sigma + it, \chi) \right] < O(\mathcal{L}) - \frac{1}{\sigma - \operatorname{Re}(\rho)}.$$

Combining this with our previous estimates yields

$$\frac{4}{\sigma + \delta - 1} < \frac{3}{\sigma - 1} + O(\mathcal{L});$$

choosing $\sigma = 1 + 4\delta$ as before yields $1/\delta \ll O(\mathcal{L})$ under the hypotheses of (3), completing the proof of (2) with the possible exception of real χ and zeros of imaginary part $\ll 1/\log q$.

Next suppose that χ is a real character and fix some $\delta > 0$, to be chosen later. We have a zero-free region for $|t| \geq \delta/\log q$. To deal with zeros of small imaginary part, let $s = \sigma$ in (1) — or, more simply, use the inequality¹ $1 + \operatorname{Re}(e^{i\theta}) \geq 0$ — to find

$$\sum_{|\operatorname{Im}(\rho)| < \delta/\log q} \operatorname{Re}(1/(\sigma - \rho)) < \frac{1}{\sigma - 1} + O(\log q),$$

the implied O -constant not depending on δ . Each term $\operatorname{Re}(1/(\sigma - \rho))$ equals $\operatorname{Re}(\sigma - \rho)/|\sigma - \rho|^2$. We choose $\sigma = 1 + (2\delta/\log q)$. Then $|\operatorname{Im}(\rho)| < \frac{1}{2}(\sigma - 1) < \frac{1}{2}\operatorname{Re}(\sigma - \rho)$, and thus $|\sigma - \rho|^2 < \frac{5}{4}\operatorname{Re}(\sigma - \rho)^2$ and $\operatorname{Re}(1/(\sigma - \rho)) > \frac{4}{5}/\operatorname{Re}(\sigma - \rho)$. So,

$$\frac{4}{5} \sum_{|\operatorname{Im}(\rho)| < \delta/\log q} \left(1 - \operatorname{Re}(\rho) + \frac{2\delta}{\log q} \right)^{-1} < \frac{\log q}{2\delta} + A \log q,$$

for some constant A independent of the choice of δ . Therefore, if δ is small enough, we can find $c > 0$ such that at most one ρ can have real part greater than $1 - (c/\log q)$. (Specifically, we may choose any $\delta < 3/10A$, and take $c = 2\delta(3 - 10A\delta)/5(1 + 2A\delta)$.) Since ρ 's are counted with multiplicity and come in complex conjugate pairs, it follows that this exceptional zero, if it exists, is real and simple. $\square$

This exceptional zero is usually denoted by β . Of course we expect, by the Extended Riemann Hypothesis, that there is no such β . The nonexistence of β , though much weaker than ERH, has yet to be proved; but we can still obtain some strong restrictions on how β can vary with q and χ . We begin by showing

¹That is, use the positivity of $-\zeta'_\chi/\zeta_\chi$, where $\zeta_\chi(s) = \zeta(s)L(s, \chi)$ is the zeta function of the quadratic number field corresponding to χ .

that at most one of the Dirichlet characters mod q can have an L -series with an exceptional zero, and deduce a stronger estimate on the error terms in our approximate formulas for $\psi(x, a \bmod q)$ and $\pi(x, a \bmod q)$. Since χ need not be primitive, it follows that in fact β cannot occur even for characters of different moduli if we set the threshold low enough:

Theorem. [Landau 1918] *There is a constant $c > 0$ such that, for any distinct primitive real characters χ_1, χ_2 to (not necessarily distinct) moduli q_1, q_2 at most one of $L(s, \chi_1)$ and $L(s, \chi_2)$ has an exceptional zero $\beta > 1 - c/\log q_1 q_2$.*

Proof: Since χ_1, χ_2 are distinct primitive real characters, their product $\chi_1 \chi_2$, while not necessarily primitive, is also a nontrivial Dirichlet character, with modulus at most $q_1 q_2$. Hence $-(L'/L)(\sigma, \chi_1 \chi_2) < O(\log q_1 q_2)$ for $\sigma > 1$. Let²

$$F(s) = \zeta(s)L(s, \chi_1)L(s, \chi_2)L(s, \chi_1 \chi_2) \quad (4)$$

Then $-F'/F$ is the sum of the negative logarithmic derivatives of $\zeta(s)$, $L(s, \chi_1)$, $L(s, \chi_2)$, and $L(s, \chi_1 \chi_2)$, which is the positive Dirichlet series

$$\sum_{n=1}^{\infty} (1 + \chi_1(n))(1 + \chi_2(n))\Lambda(n)n^{-s}.$$

In particular, this series is positive for real $s > 1$. Arguing as before, we find that if β_i are exceptional zeros of $L(s, \chi_i)$ then

$$\frac{1}{\sigma - \beta_1} + \frac{1}{\sigma - \beta_2} < \frac{1}{\sigma - 1} + O(\log q_1 q_2);$$

if $\beta_i > 1 - \delta$ then we may take $\sigma = 1 + 2\delta$ to find $1/6\delta < O(\log q_1 q_2)$, whence $\delta \gg 1/\log q_1 q_2$ as claimed. $\square$

In particular, for each q there is at most one real character mod q whose L -series has an exceptional zero $\beta > 1 - (c/\log q)$. This lets us obtain error terms that depend explicitly on q and (if it exists) β in the asymptotic formulas for $\psi(x, a \bmod q)$ and $\pi(x, a \bmod q)$. For instance (see e.g. Chapter 20 of [Davenport 1967]), we have:

Theorem. *For every $C > 0$ there exists $c > 0$ such that whenever $\gcd(a, q) = 1$ we have*

$$\psi(x, a \bmod q) = \left(1 + O(\exp -c\sqrt{\log x})\right) \frac{x}{\varphi(q)} \quad (5)$$

and

$$\pi(x, a \bmod q) = \left(1 + O(\exp -c\sqrt{\log x})\right) \frac{\text{li}(x)}{\varphi(q)} \quad (6)$$

²If $\chi_1 \chi_2$ is itself primitive then $F(s)$ is the zeta function of a biquadratic number field K , namely the compositum of the quadratic fields corresponding to χ_1 and χ_2 . In general $\zeta_K(s) = \zeta(s)L(s, \chi_1)L(s, \chi_2)L(s, \chi_3)$ where χ_3 is the primitive character underlying $\chi_1 \chi_2$; thus $F(s)$ always equals $\zeta_K(s)$ multiplied by a finite Euler product.

for all $x > \exp(C \log^2 q)$, unless there is a Dirichlet character $\chi \bmod q$ for which $L(s, \chi)$ has an exceptional zero β , in which case

$$\psi(x, a \bmod q) = \left(1 - \frac{\chi(a)x^{\beta-1}}{\beta} + O(\exp -c\sqrt{\log x})\right) \frac{x}{\varphi(q)} \quad (7)$$

and

$$\pi(x, a \bmod q) = \frac{1}{\varphi(q)} \left(\text{li}(x) - \chi(a) \text{li}(x^\beta) + O(x \exp -c\sqrt{\log x}) \right) \quad (8)$$

for all $x > \exp(C \log^2 q)$. The constant implicit in each $O(\cdot)$ may depend on C but not on q .

Proof: See the Exercises. $\square$

Just how close can this β come to 1? We first show that very small $1 - \beta$ imply small $L(1, \chi)$. Since $L(1, \chi) = \int_\beta^1 L'(\sigma, \chi) d\sigma$, it is enough to prove an upper bound on $|L'(\sigma, \chi)|$ for σ near 1. We show:

Lemma. *There exists an absolute constant C such that $|L'(\sigma, \chi)| < C \log^2 q$ for any nontrivial Dirichlet character $\chi \bmod q$ and any $\sigma \leq 1$ such that $1 - \sigma \leq 1/\log q$.*

Proof: We may assume $q > 2$, so that the series $\sum_{n=1}^\infty \chi(n)(\log n)n^{-\sigma}$ for $-L'(\sigma, \chi)$ converges if $1 - \sigma \leq 1/\log q$. Split this sum into $\sum_{n \leq q} + \sum_{n > q}$. The first sum is $O(\log^2 q)$, because the n -th term has absolute value at most

$$\frac{n^{1-\sigma}}{n} \log n \leq \frac{q^{1-\sigma}}{n} \log n \leq \frac{e}{n} \log n.$$

The sum over $n > q$ can be bounded by partial summation together with the crude estimate $|\sum_q^N \chi(n)| < q$, yielding an upper bound $e \log q$, which is again $O(\log^2 q)$. $\square$

Corollary. *If for some Dirichlet character $\chi \bmod q$ the L -series $L(s, \chi)$ has a zero $\beta > 1 - (1/\log q)$ then $L(1, \chi) < C(1 - \beta) \log^2 q$.*

But the Dirichlet class number formula for the quadratic number field corresponding to χ gives $L(1, \chi) \gg q^{-1/2}$. (We shall soon prove this directly.) Therefore

$$1 - \beta \gg \frac{1}{q^{1/2} \log^2 q}.$$

Siegel [1935] proved a much better inequality:

Theorem. *For each $\epsilon > 0$ there exists $C_\epsilon > 0$ such that*

$$L(1, \chi) > C_\epsilon q^{-\epsilon}$$

holds for all real Dirichlet characters $\chi \bmod q$. Hence there exists $C'_\epsilon > 0$ such that any zero β of $L(s, \chi)$ satisfies

$$1 - \beta > C'_\epsilon q^{-\epsilon}.$$

Proof: Let χ_1, χ_2 be different primitive real characters to moduli $q_1, q_2 > 1$, and let

$$\lambda = L(1, \chi_1)L(1, \chi_2)L(1, \chi_1\chi_2) = (s-1)F(s)\big|_{s=1},$$

with $F(s)$ as in (4). We shall prove that there exist universal constants $\theta < 1$ and $A, B, C > 0$ such that

$$F(s) > A - \frac{B\lambda}{1-s}(q_1q_2)^{C(1-s)} \quad (9)$$

holds for all $s \in (\theta, 1)$. (Specifically, we can use $\theta = 9/10$ and $A = 1/2$, $C = 8$.) Assume (9) for the time being. Since $F(s)$ is positive for $s > 1$ and has a simple pole at $s = 1$, we have $F(\beta) \leq 0$ for any $\beta \in (\theta, 1)$ such that F has no zero in $(\theta, 1)$. Of course $F(\beta) \leq 0$ also holds if β is a zero of F . For such β we have

$$\lambda > \frac{A}{B}(1-\beta)(q_1q_2)^{-C(1-\beta)}. \quad (10)$$

We shall fix χ_1 and β and use (10) to deduce a lower bound on $L(1, \chi_2)$ for all $\chi_2 \bmod q_2$ such that $q_2 > q_1$. If there is some real χ_1 such that $L(\beta_1, \chi_1) = 0$ for some $\beta_1 > 1 - (\epsilon/2C)$ then we use that character for χ_1 and the zero β_1 for β . Otherwise $F(s)$ never has a zero in $(1 - (\epsilon/2C), 1)$, so we choose χ_1 arbitrarily and β subject to $0 < 1 - \beta < \epsilon/2C$. Then for any primitive $\chi_2 \bmod q_2 > q_1$ we use (10), together with the upper bound $L(1, \chi) \ll \log q$ (see the Exercises), to find that

$$L(1, \chi_2) > c q_2^{-C(1-\beta)} / \log q_2,$$

with c depending only (but ineffectively!) on ϵ via χ_1 and β . Since $C(1-\beta) < \epsilon/2$, Siegel's theorem follows.

It remains to prove (9). Siegel originally showed this using class field theory; we follow the more direct approach of [Estermann 1948]. (See also [Chowla 1950] for another direct proof.)

Since $F(s)$ has a nonnegative Dirichlet series, its Taylor series about $s = 2$ is

$$F(s) = \sum_{m=0}^{\infty} b_m (2-s)^m$$

with $b_0 = F(2) > 1$ and all $b_m > 0$. Since F is entire except for a simple pole of residue λ at $s = 1$, we have the Taylor expansion

$$F(s) - \frac{\lambda}{s-1} = \sum_{m=0}^{\infty} (b_m - \lambda)(2-s)^m,$$

valid for all $s \in \mathbf{C}$. Consider this on $|s-2| = 3/2$. We have there the crude bounds $L(s, \chi_1) \ll q_1$, $L(s, \chi_2) \ll q_2$, $L(s, \chi_1\chi_2) \ll q_1q_2$, and of course $\zeta(s)$ is

bounded on $|s - 2| = 3/2$. So, $F(s) \ll (q_1 q_2)^2$ on this circle, and thus the same is true of $F(s) - \lambda/(s - 1)$. Hence

$$|b_m - \lambda| \ll (2/3)^m (q_1 q_2)^2.$$

For any fixed $\theta \in (1/2, 1)$ it follows that

$$\sum_{m=M}^{\infty} |b_m - \lambda| (2 - s)^m \ll (q_1 q_2)^2 \left(\frac{2}{3} (2 - \theta) \right)^M$$

holds for all $s \in (\theta, 1)$. Since $b_0 > 1$ and $b_m \geq 0$, we thus have

$$F(s) - \frac{\lambda}{s - 1} \geq 1 - \lambda \frac{(2 - s)^M - 1}{1 - s} - O(q_1 q_2)^2 \left(\frac{2}{3} (2 - \theta) \right)^M.$$

Let M be the largest integer such that the error estimate $O(q_1 q_2)^2 ((4 - 2\theta)/3)^M$ is $< 1/2$. Then

$$F(s) > \frac{1}{2} - \frac{\lambda}{1 - s} (2 - s)^M.$$

But

$$(2 - s)^M = \exp(M \log(2 - s)) < \exp M(1 - s),$$

and $\exp M \ll (q_1 q_2)^{O(1)}$, which completes the proof of (9) and thus of Siegel's theorem.

Remarks

Unfortunately the constants C_ϵ and C'_ϵ in Siegel's theorem, unlike all such constants that we have encountered so far, are ineffective for every $\epsilon < 1/2$, and remain ineffective almost seventy years later. This is because we need more than one counterexample to reach a contradiction. What can be obtained effectively are constants C_ϵ such that $1 - \beta > C_\epsilon q^{-\epsilon}$ holds for every q and all real primitive characters mod q , with at most a single exception (q, χ, β) . This β , if it exists, is called the ‘‘Siegel zero’’ or ‘‘Siegel-Landau zero’’. Note that a zero β of some $L(s, \chi)$ that violates the ERH may or may not qualify as a Siegel(-Landau) zero depending on the choice of ϵ and C_ϵ .

Dirichlet's class number formula relates $L(1, \chi)$ for real characters χ with the class number of quadratic number fields. Siegel's theorem and its refinements thus yield information on these class numbers. For instance, if χ is an odd character then the imaginary quadratic field $K = \mathbf{Q}(\sqrt{-q})$ has a zeta function $\zeta_K(s)$ that factors into $\zeta(s)L(s, \chi)$. Let $h(K)$ be the class number of K . Siegel's theorem, together with Dirichlet's formula, yields the estimate $h(K) \gg_\epsilon q^{1/2-\epsilon}$. In particular, $h(K) > 1$ for all but finitely many K . But this does not reduce the determination of all such K to a finite computation because the implied constant cannot be made effective. Gauss had already conjectured in 1801 (in terms of binary quadratic forms, see *Disq. Arith.*, §303) that $h(K) = 1$ only for $K = \mathbf{Q}(\sqrt{-q})$ with

$$q = 3, 4, 7, 8, 11, 19, 43, 67, 163.$$

But the closest that Siegel's method can bring us to this conjecture is the theorem that there is at most one further such q . Heilbronn and Linfoot showed this a year before Siegel's theorem [HL 1934] using a closely related method [Heilbronn 1934].

It was only with much further effort that Heegner ([1952], corrected by Deuring [1968]), and later Baker [1966] and Stark [1967] (working independently and using different approaches), proved Gauss's conjecture, at last exorcising the "tenth discriminant". None of these approaches yields an effective lower bound on $h(K)$ that grows without limit as $q \rightarrow \infty$. Such a bound was finally obtained by Goldfeld, Gross, and Zagier ([Goldfeld 1976], [GZ 1986]), but it grows much more slowly than $q^{1/2-\epsilon}$; namely, $h(K) > c \log q / \log \log q$ (and $h(K) > c' \log q$ for prime q). Even this was a major breakthrough that combined difficult algebraic and analytic techniques. See [Goldfeld 1986] for an overview.

Exercises

1. Complete the derivation of (5,6,7,8) from our (nearly) zero-free region for Dirichlet L -functions.
2. Show that for each $\epsilon > 0$ there exists C such that whenever $\gcd(a, q) = 1$ we have

$$|\varphi(q) \psi(x, a \bmod q) - x| < \epsilon x$$

for all $x > q^C$, unless there is a Dirichlet character $\chi \bmod q$ for which $L(s, \chi)$ has an exceptional zero β , in which case

$$|\varphi(q) \psi(x, a \bmod q) - x - \chi(a)(x^\beta/\beta)| < \epsilon x$$

for all $x > q^C$. [NB $q^C = \exp(C \log q)$.] Use this argument, together with the fact that $\psi(x, a \bmod q) \geq 0$, to give an alternative proof of Landau's theorem (at most one exceptional zero for any character mod q). Also, obtain the corresponding estimates for $\pi(x, a \bmod q)$, and deduce that there is an absolute constant C such that if there is no exceptional zero then there exists a prime $p \equiv a \bmod q$ with $p < q^C$. (Linnik [1944] showed this unconditionally, by showing that very small values of $1 - \beta$ force the other zeros farther away from the line $\sigma = 1$.)

3. Check that $|L'(\sigma, \chi)| \ll_A \log^2 q$ holds in any interval $0 \leq 1 - \sigma \leq A/\log q$. How does the implied constant depend on A ? Use the same method to show that also $|L(\sigma, \chi)| \ll \log q$ in the same interval, and in particular at $\sigma = 1$. What bound can you obtain on the higher derivatives of $L(\sigma, \chi)$ for σ near 1?
4. Prove that for every $\epsilon > 0$ there exist positive constants A, c such that if $L(1, \chi) < Aq^{-\epsilon}$ for some primitive Dirichlet character $\chi \bmod q$ then $L(\beta, \chi) = 0$ for some β such that $1 - \beta < c/\log q$. [Use the fact that

$$L(1, \chi) = L(\sigma, \chi) \exp \int_1^\sigma -\frac{L'}{L}(s, \chi) ds,$$

or work directly with the Hadamard product formula for $L(s, \chi)$. This Exercise, which shows that small $L(1, \chi)$ implies small $1 - \beta$, may be regarded as a

qualitative converse of our computation showing that small $1 - \beta$ implies small $L(1, \chi)$.]

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