Math 250: Higher Algebra
Representations of finite groups

1 Basic definitions

Representations. A *representation* of a group G over a field k is a k-vector space V together with an action of G on V by linear maps. Equivalently, it is a homomorphism from G to $\mathrm{GL}(V)$. This is also the same as a representation of the group algebra $k[G]$, in other words, a $k[G]$ module. A *subrepresentation* is a subspace W such that $gw \in W$ for all $g \in G$ and $w \in W$. A representation V is *irreducible* if $V \neq \{0\}$ and the only subrepresentations of V are $\{0\}$ and V itself. It is *trivial* if $gv = v$ for all $g \in G$ and $v \in V$. For instance, any representation V has the trivial subrepresentation $V^G := \{v \in V : \forall g \in G, gv = v\}$.

Operations on representations. Let V, W be representations of G. The direct sum $V \oplus W$ is then also a representation, with $g((v,w)) = (gv, gw)$. More interestingly, we can consider vector spaces of homomorphisms from V to W. We define $\mathrm{Hom}_G(V,W)$ to be the vector space of $k[G]$-linear maps from V to W. These are the k-linear maps $T : V \to W$ that commute with the action of G: for all $v \in V$, we must have $g(Tv) = T(gv)$. We can then define $\mathrm{End}_G(V) = \mathrm{Hom}_G(V,V)$. We make $\mathrm{Hom}_k(V,W)$ into a representation of G by setting $g(T) = g \circ T \circ g^{-1}$. Note that

$$\mathrm{Hom}_G(V,W) = (\mathrm{Hom}_k(V,W))^G.$$

In particular, if $W = k$ then $\mathrm{Hom}_k(V,W) = V^*$, so we have given V^* the structure of a representation of G, called the *contragredient* of V. Consistent with our definition of the action of G on $\mathrm{Hom}_k(V,W)$, we make $V \otimes_k W$ into a representation of G by defining $g(v \otimes w) = gv \otimes gw$ and extending by linearity.

Characters. The center of $k[G]$ consists of the *class functions*, that is, functions constant on conjugacy classes of G (so $f(g) = f(hgh^{-1})$ for all $g, h \in G$). An important example of a class function is the *character* of a finite-dimensional representation V. This is the map $\chi = \chi_V : G \to k$ taking each $g \in G$ to the trace of the linear map $g : V \to V$. For instance, $\chi(1) = \dim V$, and if V is trivial then $\chi(g) = \dim V$ for all $g \in G$. If V is a permutation representation then $\chi(g)$ is the number of fixed points of g. An important special case is the permutation representation arising from the action of G on itself by left multiplication. This recovers the (left) regular representation $k[G]$, whose character is $\chi(1) = |G|$ and $\chi(g) = 0$ for all $g \neq 1$.

If V, W are finite dimensional then so are $V \oplus W$ and $V \otimes_k W$, and their characters are given by

$$\chi_{V \oplus W} = \chi_V + \chi_W, \quad \chi_{V \otimes_k W} = \chi_V \chi_W.$$

If $k \subseteq \mathbf{C}$ and G is finite (or more generally consists of elements of finite order) then each $g \in G$ acts by a linear transformations all of whose eigenvalues λ are roots of unity. Hence $\chi(g)$ is an algebraic integer contained in some cyclotomic extension of $\mathbf{Q}$. Moreover, $\lambda^{-1} = \overline{\lambda}$. Hence the character of the contragredient representation is given by

$$\chi_{V^*}(g) = \overline{\chi_V(g)} = \chi_V(g^{-1}).$$

2 Theorems

Throughout this section, G is a finite group, and k is a field whose characteristic is not a factor of G.

Theorem 1. *The ring $k[G]$ is a finite direct sum of simple k-algebras.*

By Wedderburn, each of these simple algebras is $\mathrm{End}_K V$ for some skew field $K \supseteq k$ and some K-vector space V, with $\dim_k K$ and $\dim_K V$ finite. Each of these V is an irreducible representation of G over k.

Theorem 2. *Every representation of G is a direct sum of irreducible representations, each of which is isomorphic to one of those V. Each V occurs in the left regular representation with multiplicity* $\dim_K V$.

In particular, if k is algebraically closed then each K is k. Then k is the center of $\mathrm{End}(V)$. Since the center of $k[G]$ is the direct sum of these centers, we can compare dimensions to obtain:

Corollary 1. *If k is algebraically closed then the number of isomorphism classes of irreducible representations V of G equals the number of conjugacy classes of G, and the sum of* $(\dim V)^2$ *over those representations equals $|G|$.*

We can also describe the irreducible representations of a product of two groups. If V, W are representations of G, H respectively, then $G \times H$ acts on $V \otimes W$ by $(g, h)(v \otimes w) = gv \otimes hw$.

Corollary 2. *Let V, W be irreducible representations of finite groups G, H over an algebraically closed field k whose characteristic divides neither $|G|$ nor $|H|$. Then $V \otimes W$ is an irreducible representation of $G \times H$, and every irreducible representation of $G \times H$ arises in this way for a unique choice of V and W.*

Of course, over any field the irreducible representations of a quotient group G/H are just the irreducible representations of G whose restriction to H is trivial.

Suppose now that $k \subseteq \mathbf{C}$. Define a sesquilinear inner product $\langle \cdot, \cdot \rangle$ on functions on G:

$$\langle f_1, f_2 \rangle := \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}.$$

Theorem 3. *For any finite-dimensional representation V of G with character χ, we have* $\dim V^G = \langle \chi, 1 \rangle$. *If k is algebraically closed, and W is any irreducible representation of G, then the multiplicity of W in V is $\langle \chi, \chi_W \rangle$.*

If k is not algebraically closed, then a similar formula holds but we must divide $\langle \chi, \chi_W \rangle$ by the dimension of the center of the skew field K corresponding to W.

Theorem 4. *If k is an algebraically closed subfield of $\mathbf{C}$ then the characters of the irreducible representations of G constitute an orthonormal basis for the class functions on G.*

Corollary. *If k is an algebraically closed subfield of $\mathbf{C}$, and V is a representation of G over k of positive finite dimension, then $\langle \chi_V, \chi_V \rangle$ is a positive integer, which equals 1 if and only if the representation is irreducible.*

If $\langle \chi_V, \chi_V \rangle = 2$ or 3 then V is the sum of two or three different irreducible representations, but usually some work is needed to obtain them.

Theorem 5. *Assume that k is an algebraically closed subfield of $\mathbf{C}$, and let g, h be non-conjugate elements of G. Then the sum of $\chi(g)\overline{\chi(h)}$ over characters χ of irreducible representations of G vanishes. If g, h are conjugate, then the sum of $\chi(g)\overline{\chi(h)} = |\chi(g)|^2$ is the order of the centralizer $\{x \in G : xg = gx\}$ of g.*

Recall that the centralizer order is also the quotient of $|G|$ by the size of the conjugacy class. More generally, for any conjugacy classes $[g_1], \ldots, [g_m]$ one may enumerate solutions of $x_1 \cdots x_m = 1$ in $x_i \in [g_i]$ using a weighted sum of $\chi(g_1) \cdots \chi(g_m)$ over irreducible characters; see for instance the section on the "rigidity method" in Serre's *Topics in Galois Theory.*

3 Proofs

Idempotents. A homomorphism $\alpha : A \to A$ from any abelian group A to itself is said to be an *idempotent* if $\alpha^2 = \alpha$. For example, the e_i in Lang's proof of the decomposition of a semisimple ring (*Algebra*, Chapter XVII, §4) are idempotents. If α is an idempotent then so is $1 - \alpha$, the *complementary idempotent.* Its kernel is the image of α and vice versa; A is the direct sum of αA and $(1-\alpha)A$. Conversely any direct sum decomposition $A = A_1 \oplus A_2$ comes from a complementary pair of idempotents, namely the projections of A to A_1 and A_2.

The averaging idempotent; semisimplicity of $k[G]$. Now let G be a finite group, and k a field whose characteristic does not divide $|G|$. Then $k[G]$ contains the idempotent[1]

$$\alpha := \frac{1}{|G|} \sum_{g \in G} g.$$

This is a central element of $k[G]$; indeed $\alpha g = g\alpha = g$ for all $g \in G$. It follows that if V is any representation of G then $V^G = \alpha G$.

[1] Note the factor of $|G|$ in the denominator. This explains our assumption about $\operatorname{char}(k)$. If $|G|$ is a multiple of the characteristic, then instead of a central idempotent we have a central *nilpotent* element $\sum_{g \in G} g$, and then $k[G]$ cannot be semisimple. Much is still known about the representations of G in this case, but the theory is considerably subtler, and at least for the present we shall not delve into it.

We can use α to show that $k[G]$ is semisimple (i.e., that every $k[G]$ module is semisimple), again assuming that $|G|$ is not a multiple of char(k). Let V be any $k[G]$ module, and V' any submodule. Let $p : V \to V'$ be any linear map whose restriction to the identity is V'. (For instance, choose a complementary subspace V'', so $V = V' \oplus V''$, and let $p : V \to V'$ be the projection map). In general this need not be a $k[G]$-module homomorphism. However,

$$\pi := \alpha p = \frac{1}{|G|} \sum_{g \in G} gpg^{-1}$$

is a homomorphism of $k[G]$ modules. Moreover, the restriction of π to V' is still the identity. Then V is the direct sum of the $k[G]$ modules V' and $\ker \pi$. Since V' was an arbitrary submodule, we have proved that V is semisimple. Thus $k[G]$ is semisimple as claimed.

Decomposition of $k[G]$ and irreducible representations of G. It follows (Lang, *loc.cit.*) that $k[G]$ is a finite direct sum of simple rings, one for each representation of $k[G]$. This proves Theorem 1.

By Wedderburn, each of these simple rings is of the form $M_n(K)$ for some (possibly skew) field K containing k, corresponding to an irreducible representation of $k[G]$ on an n-dimensional vector space V over K. This, together with general facts about semisimple rings, proves Theorem 2.

By comparing dimensions we see that $|G|$ is the sum of $n^2 \dim_k K$ extended over these representations. In particular, if k is algebraically closed then $K = k$ and

$$|G| = \sum_V (\dim V)^2,$$

as noted in Corollary 1 to Theorem 2. In this case, $k[G \times H] = k[G] \otimes k[H]$ is a direct sum of tensor products of matrix algebras $\text{End}(V) \otimes \text{End}(W)$; since $\text{End}(V) \otimes \text{End}(W) = \text{End}(V \otimes W)$, we recover Corollary 2 to Theorem 2.

If $k = \mathbf{R}$ then $\text{End}_G(V)$ is either $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$; we naturally call such representations "real", "complex", or "quaternionic" respectively. For instance, $\mathbf{R}$, $\mathbf{C}$ and $\mathbf{H}$ are themselves representations of the finite groups $\{\pm 1\}$, $\{\pm 1, \pm i\}$, and $\{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}$ which are irreducible over $\mathbf{R}$.

Consequences of Schur's lemma. In our context, Schur's lemma says that if V, W are irreducible representations of G then every G-homomorphism from V to W is either zero or an isomorphism. It follows (Jordan-Hölder) that if a representation of G is written in two ways as a finite direct sum of irreducibles, say $\oplus_{i=1}^m V_i \cong \oplus_{j=1}^n W_j$, then $m = n$ and the V_i are some permutation of the W_j.

Now let V be an irreducible representation of G, and consider $\text{End}_G(V) = \text{Hom}_G(V, V)$. By Schur, each nonzero element of this k-algebra is invertible. That is, $\text{End}_G(V)$ is a division algebra containing k. This is just the skew field K we associated to V earlier.

Characters and orthogonality relations. If V is a nontrivial irreducible representation then $\alpha V = 0$. In particular, the image of α in End_V has zero trace. If V is the trivial irreducible representation of dimension 1 then of course the trace of the image of α in End_V is 1. Since every finite-dimensional representation V is the direct sum of irreducibles, and the trace is additive, we conclude that

$$\dim V^G = \mathrm{tr}(\alpha|_V) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g).$$

(Warning: if k has characteristic $p > 0$ then this formula determines $\dim V^G$ only up to a multiple of p.)

Suppose now that $k = \mathbf{C}$ (more generally, that k is an algebraically closed subfield of $\mathbf{C}$). Let V be an irreducible representation of G, and W any finite-dimensional representation. Then $\dim \mathrm{Hom}_G(W, V)$ is the multiplicity of V in the decomposition of W into irreducibles. But

$$\mathrm{Hom}_G(W, V) = (\mathrm{Hom}_k(W, V))^G = (W^* \otimes V)^G.$$

Hence the dimension of $\mathrm{Hom}_G(W, V)$ is

$$\frac{1}{|G|} \sum_{g \in G} \chi_{V \otimes W^*}(g) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \chi_{W^*}(g) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \overline{\chi_W(g)} = \langle \chi_V, \chi_W \rangle.$$

This proves Theorem 3.

In particular, if we take V to be irreducible as well, we deduce the orthonormality of the characters of irreducible representations of G. Each of these characters is a class function, and by the Corollary to Theorem 2 their number equals the dimension of the space of class functions. This proves Theorem 4.

We may also take W to be the regular representation $k[G]$. We then find that the multiplicity of V in this representation is $\chi_V(1) = \dim V$. Comparing dimensions, we find again that

$$|G| = \dim k[G] = \sum_V \dim(V^{\dim V}) = \sum_V (\dim V)^2,$$

We obtained this by in effect taking $\chi(1)$ on both sides of $k[G] = \oplus_V V^{\dim V}$. If we instead take $\chi(g)$ for some $g \neq 1$, we find $\sum_V \dim(V) \chi_V(g) = 0$. This is the special case $h = 1$ of Theorem 5. To prove all of Theorem 5 in this way, we may let $G \times G$ act on G by $(g, h)x = gxh^{-1}$, and thence on $k[G]$ as a permutation representation. The simple factors of $k[G]$ become irreducible representations $V \otimes V^*$ of $G \otimes G$, with characters $\chi_V(g)\overline{\chi_W(h)}$. If g, h are not conjugate in G, then $x \mapsto gxh^{-1}$ has no fixed points, so the action of (g, h) on $k[G]$ has trace zero. If g, h are conjugate then $\#\{x \in G : x = gxh^{-1}\}$ is the size of the centralizer of g. This proves Theorem 5.

Theorem 5 could also be proved directly from Theorem 4 using the fact that a square matrix with orthogonal rows also has orthogonal columns.