

**Math 250: Higher Algebra**

Problem Set #5 (29 October 2004): Representations of finite groups

First, two exercises from Fulton-Harris on induced representations. Let  $G$  be a finite group, and  $H$  any subgroup.

1. [Fulton-Harris, Exercise 3.16 on page 34]

i) If  $U$  is a representation of  $G$  and  $W$  is a representation of  $H$ , show that

$$U \otimes \text{Ind}W = \text{Ind}(\text{Res}(U) \otimes W).$$

In particular,  $\text{Ind}(\text{Res}(U)) = U \otimes P$ , where  $P$  is the permutation representation of  $G$  on  $G/H$ .

ii) If  $H \subset K \subset G$  with  $K$  also a subgroup, show that

$$\text{Ind}_H^G W = \text{Ind}_K^G (\text{Ind}_H^K W)$$

(so induction, like restriction, is transitive).

2. [Fulton-Harris, Exercise 3.19 on page 34] Let  $W$  be a finite-dimensional representation of  $H$ , and  $V = \text{Ind}_H^G W$ . Quoting Fulton-Harris (p.34):

To compute [its] character, note that  $g \in G$  maps  $\sigma W$  to  $g\sigma W$ , so the trace is calculated from those cosets  $\sigma$  with  $g\sigma = \sigma$ , i.e.,  $s^{-1}gs \in H$  for [any, equivalently all]  $s \in \sigma$ . Therefore,

$$\chi_{\text{Ind}W}(g) = \sum_{g\sigma=\sigma} \chi_W(s^{-1}gs) \quad (s \in \sigma \text{ arbitrary}). \quad (3.18)$$

i) If  $C$  is the conjugacy class of  $g$  in  $G$ , and  $C \cap H$  decomposes into conjugacy classes  $D_1, \dots, D_r$  of  $H$ , then (3.18) can be rewritten as:

$$\chi_{\text{Ind}W}(g) = [G : H] \sum_{i=1}^r \frac{|D_i|}{|C|} \chi_W(h_i)$$

for  $h_i \in D_i$ .

ii) If  $W$  is the trivial representation of  $H$ , then

$$\chi_{\text{Ind}W}(g) = \frac{[G : H]}{|C|} |C \cap H|.$$

iii) Deduce the Frobenius reciprocity formula (Cor. 3.20 on p.35): for any finite-dimensional representations  $U, W$  of  $G, H$  respectively,

$$(\chi_{\text{Ind}W}, \chi_U)_G = (\chi_W, \chi_{\text{Res}U})_H.$$

We next describe “extraspecial 2-groups” and their representations. The 8-element dihedral and quaternion groups are the familiar first examples of extraspecial 2-groups.

3. Let  $G$  be a finite group whose center contains a 2-element group  $\{\pm 1\}$ , and let  $A$  be the quotient group  $G/\{\pm 1\}$ . Let  $V$  be any irreducible representation of  $G$ , of dimension  $d$ . Show that the restriction of  $V$  to  $\{\pm 1\}$  consists of either  $d$  copies of the trivial representation — in which case it comes from a representation of  $A$  — or  $d$  copies of the nontrivial 1-dimensional representation of  $\{\pm 1\}$ .

Equivalently,  $-1 \in G$  acts on  $V$  by multiplication by a scalar, which is necessarily either  $+1$  or  $-1$ . We'll call these two kinds of representations of  $G$  “even” and “odd” respectively.

4. Now suppose  $A$  is an “elementary abelian 2-group”, that is, a group isomorphic with  $(\mathbf{Z}/2\mathbf{Z})^m$  for some  $m$ . Define a map  $(\cdot, \cdot) : A \times A \rightarrow \{\pm 1\}$  as follows: for any  $a, b \in A$ , let  $g \in G$  be either of the preimages of  $a$ , and let  $h \in G$  be either of the preimages of  $b$ ; then  $(a, b)$  is the commutator  $ghg^{-1}h^{-1}$ . Explain why  $(a, b)$  is in fact in  $\{\pm 1\}$  and is well-defined (independent of the choice of  $g, h$ ). Then show that  $(\cdot, \cdot)$  is bilinear and alternating, i.e., that it satisfies the identities

$$(aa', b) = (a, b)(a', b), \quad (a, bb') = (a, b)(a, b'), \quad (a, a) = 1, \quad (a, b) = (b, a).$$

Prove that this pairing is nondegenerate if and only if  $\{\pm 1\}$  is the center of  $G$ , and if and only if each  $g \in G - \{\pm 1\}$  is conjugate to  $-g = (-1)g$ . In this case,  $G$  is said to be an “extraspecial 2-group”. Note that the 8-element dihedral and quaternion groups are indeed extraspecial 2-groups with  $m = 2$ .

5. Now let  $G$  an extraspecial 2-group of order  $2^{m+1}$ .
- i) Show that  $G$  has  $2^m$  even representations, each of dimension 1. Deduce that  $G$  has a unique odd representation  $V$  and that  $V$  has dimension  $2^{m/2}$ . [In particular it follows that  $m$  must be even — which we could also obtain from the nondegeneracy of the alternating form  $(\cdot, \cdot)$ .] Determine the character  $\chi_V$  of this representation, and note that  $\chi_V(g) \in \mathbf{R}$  for all  $g \in G$ , whence  $V$  is necessarily either real or quaternionic.
  - ii) For  $a \in A$  define  $Q(a) = g^2$  where  $g$  is either of the preimages of  $a$  in  $G$ . Explain why this gives a well-defined map from  $A$  to  $\{\pm 1\}$ , and show that it is a quadratic form whose associated bilinear form is  $(\cdot, \cdot)$  (which, in our multiplicative notation for the  $G$  and its center, means that  $(a, b) = Q(ab)/(Q(a)Q(b))$  for all  $a, b \in A$ ). Use the formula of Exercise 3.38\* (p.41) to conclude that  $V$  is real or quaternionic according as  $\sum_{a \in A} Q(a)$  equals  $2^{m/2}$  or  $-2^{m/2}$ .

Note that this confirms the known behavior of the 8-element extraspecial 2-groups. It turns out that any quadratic form on  $(\mathbf{Z}/2\mathbf{Z})^m$  comes from some extraspecial 2-group, and thus in particular satisfies  $\sum_{a \in A} Q(a) = \pm 2^{m/2}$ ; and that two quadratic forms on  $(\mathbf{Z}/2\mathbf{Z})^m$  are equivalent under  $\text{Aut}((\mathbf{Z}/2\mathbf{Z})^m) = \text{GL}_{2m}(\mathbf{Z}/2\mathbf{Z})$  if and only if their invariants  $2^{-m/2} \sum_{a \in A} Q(a)$  are equal. It follows that for each positive integer  $m$  there are two extraspecial 2-groups of order  $2^{m+1}$ , usually denoted  $2_+^{1+m}$  and  $2_-^{1+m}$ . For instance, the 8-element dihedral and quaternion groups are  $2_+^{1+2}$  and  $2_-^{1+2}$ .

Problem set is due in class Friday the 5th of November.