Math 229: Introduction to Analytic Number Theory

A zero-free region for $\zeta(s)$

We first show, as promised, that $\zeta(s)$ does not vanish on $\sigma = 1$. As usual nowadays, we give Mertens' elegant version of the original arguments of Hadamard and (independently) de la Vallée Poussin. Recall that

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

has a simple pole at $s = 1$ with residue $+1$. If $\zeta(s)$ were to vanish at some $1 + it$ then $-\zeta'/\zeta$ would have a simple pole with residue -1 (or $-2, -3, \ldots$) there. The idea is that $\sum_n \Lambda(n)/n^s$ converges for $\sigma > 1$, and as s approaches 1 from the right all the terms contribute towards the positive-residue pole. As $\sigma \to 1 + it$ from the right, the corresponding terms have the same magnitude but are multiplied by n^{-it}, so a pole with residue -1 would force "almost all" the phases n^{-it} to be near -1. But then near $1 + 2it$ the phases n^{-2it} would again approximate $(-1)^2 = +1$, yielding a pole of positive residue, which is not possible because then ζ would have another pole besides $s = 1$.

To make precise the idea that if $n^{-it} \approx -1$ then $n^{-2it} \approx +1$, we use the identity

$$2(1 + \cos\theta)^2 = 3 + 4\cos\theta + \cos 2\theta,$$

from which it follows that the right-hand side is positive. Thus if $\theta = t \log n$ we have

$$3 + 4\,\mathrm{Re}(n^{-it}) + \mathrm{Re}(n^{-2it}) \geq 0.$$

Multiplying by $\Lambda(n)/n^\sigma$ and summing over n we find

$$3\left[-\frac{\zeta'}{\zeta}(\sigma)\right] + 4\,\mathrm{Re}\left[-\frac{\zeta'}{\zeta}(\sigma + it)\right] + \mathrm{Re}\left[-\frac{\zeta'}{\zeta}(\sigma + 2it)\right] \geq 0 \tag{1}$$

for all $\sigma > 1$ and $t \in \mathbf{R}$. Fix $t \neq 0$. As $\sigma \to 1+$, the first term in the LHS of this inequality is $3/(\sigma - 1) + O(1)$, and the remaining terms are bounded below. If ζ had a zero of order $r > 0$ at $1 + it$, the second term would be $-4r/(\sigma - 1) + O(1)$. Thus the inequality yields $4r \leq 3$. Since r is an integer, this is impossible, and the proof is complete.

We next use (1), together with the partial-fraction formula

$$-\frac{\zeta'}{\zeta}(s) = \frac{1}{s-1} + B_1 + \frac{1}{2}\frac{\Gamma'}{\Gamma}(\frac{s}{2} + 1) - \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right),$$

to show that even the existence of a zero close to $1 + it$ is not possible. How close depends on t; specifically, we show:[1]

[1]See for instance Chapter 13 of Davenport's book [Davenport 1967] cited earlier. This classical bound has been improved; the current record of $1 - \sigma \ll \log^{-2/3-\epsilon}|t|$, due to Korobov and perhaps Vinogradov, has stood for 50 years. See [Walfisz 1963] or [Montgomery 1971, Chapter 11].

Theorem. *There is a constant $c > 0$ such that if $|t| > 2$ and $\zeta(\sigma + it) = 0$ then*

$$\sigma < 1 - \frac{c}{\log |t|}. \tag{2}$$

Proof: Let $\sigma \in [1, 2]$ and[2] $|t| \geq 2$ in the partial-fraction formula. Then the B_1 and Γ'/Γ terms are $O(\log |t|)$, and each of the terms $1/(s - \rho)$, $1/\rho$ has positive real part as noted in connection with von Mangoldt's theorem on $N(T)$. Therefore[3]

$$-\operatorname{Re} \frac{\zeta'}{\zeta}(\sigma + 2it) < O(\log |t|),$$

and if some $\rho = 1 - \delta + it$ then

$$-\operatorname{Re} \frac{\zeta'}{\zeta}(\sigma + it) < O(\log |t|) - \frac{1}{\sigma + \delta - 1}.$$

Thus (1) yields

$$\frac{4}{\sigma + \delta - 1} < \frac{3}{\sigma - 1} + O(\log |t|).$$

In particular, taking[4] $\sigma = 1 + 4\delta$ yields $1/20\delta < O(\log |t|)$. Hence $\delta \gg (\log |t|)^{-1}$, and our claim (2) follows. $\Box$

Once we obtain the functional equation and partial-fraction decomposition for Dirichlet L-functions $L(s, \chi)$, the same argument will show that (2) also gives a zero-free region for $L(s, \chi)$, though with the implied constant depending on χ.

Remarks

The only properties of $\Lambda(n)$ that we used in the proof of $\zeta(1 + it) \neq 0$ are that facts that $\Lambda(n) \geq 0$ for all n and that $\sum_n \Lambda(n)/n^s$ has an analytic continuation with a simple pole at $s = 1$ and no other poles of real part ≥ 1. Thus the same argument exactly will show that $\prod_{\chi \bmod q} L(s, \chi)$, and thus each of the factors $L(s, \chi)$, has no zero on the line $\sigma = 1$.

The $3 + 4\cos\theta + \cos 2\theta$ trick is worth remembering, since it has been adapted to other uses. For instance, we shall revisit and generalize it when we develop the Drinfeld-Vlăduţ upper bounds on points of a curve over a finite field and the Odlyzko-Stark lower bounds on discriminants of number fields. See also the following Exercises.

[2] A lower bound $|t| \geq t_0$ would do for any $t_0 > 1$ — and the only reason we cannot go lower is that our bounds are in terms of $\log |t|$ so we do not want to allow $\log |t| = 0$.

[3] Note that we write $< O(\log |t|)$, not $= O(\log |t|)$, to allow the possibility of an arbitrarily large *negative* multiple of $|\log t|$.

[4] $1 + \alpha\delta$ will do for any $\alpha > 3$. This requires that $\alpha\delta \leq 1$, e.g. $\delta \leq 1/4$ for our choice of $\alpha = 4$, else $\sigma > 2$; but we're concerned only with δ near zero, so this does not matter.

Exercises

1. Use the inequality $3 + 4\cos\theta + \cos 2\theta \geq 0$ to give an alternative proof that $L(1,\chi) \neq 0$ when χ is a complex Dirichlet character (a character such that $\chi \neq \overline{\chi}$).

2. Show that for each $\alpha > 2$ there exists $t \in \mathbf{R}$ such that

$$\int_{-\infty}^{\infty} \exp(-|x|^{\alpha} + itx)\, dx < 0.$$

(Yes, this is related to the present topic; see [EOR 1991, p.633]. The integral is known to be positive for all $t \in \mathbf{R}$ when $\alpha \in (0, 2]$; see for instance [EOR 1991, Lemma 5].)

References

[EOR 1991] Elkies, N.D., Odlyzko, A., Rush, J.A.: On the packing densities of superballs and other bodies, Invent. Math. 105 (1991), 613–639.

[Montgomery 1971] Montgomery, H.L.: *Topics in Multiplicative Number Theory.* Berlin: Springer, 1971. [LNM **227** / QA3.L28 #227]

[Walfisz 1963] Walfisz, A.: *Weylsche Exponentialsummen in der neueren Zahlentheorie.* Berlin: Deutscher Verlag der Wissenschaften, 1963. [AB 9.63.5 / Sci 885.110(15,16)]