Math 229: Introduction to Analytic Number Theory

The product formula for $\xi(s)$ and $\zeta(s)$; vertical distribution of zeros

Behavior on vertical lines. We next show that $(s^2 - s)\xi(s)$ is an entire function of order 1; more precisely:

Lemma. *There exists a constant C such that $(s^2-s)\xi(s) \ll \exp(C|s|\log|s|)$, but no constant C' such that $(s^2-s)\xi(s) \ll \exp(C'|s|)$.*

Proof: The second part is easy, because $\xi(s)$ already grows faster than $\exp(C'|s|)$ as $s \to \infty$ along the positive real axis. Indeed for $s > 1$ we have $\zeta(s) > 1$, while the factor $(s^2-s)\pi^{-s/2}\Gamma(s/2)$ of $(s^2-s)\xi(s)$ grows faster than $\exp(C'|s|)$ by Stirling. For the first part, since $|s|\log|s|$ is bounded below we need only give an upper bound on $(s^2-s)\xi(s)$ for large $|s|$, and the functional equation $\xi(s) = \xi(1-s)$ lets us assume that $s = \sigma + it$ with $\sigma \geq 1/2$. Moreover, the sum and integral formulas for $\zeta(s)$ and $\Gamma(s)$ yield $|\xi(\sigma+it)| \leq \pi^{-\sigma/2}\Gamma(\sigma/2)\zeta(\sigma)$ whenever $\sigma > 1$. By Stirling we readily conclude that $(s^2-s)\xi(s) \ll \exp(C|s|\log|s|)$ holds if s is far enough from the critical strip, say in the half-plane $\sigma \geq 2$. To extend this bound to $\sigma \geq 1/2$, it will be more than enough to prove that $\zeta(s) \ll |t|$ for $\sigma \geq 1/2$ and $|t| \geq 1$; and this follows from our formula

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=1}^{\infty} \int_n^{n+1} (n^{-s} - x^{-s})\,dx$$

for $\zeta(s)$ as a meromorphic function on the half-plane $\sigma > 0$, in which we saw that the sum is $O(|s|)$. $\square$

Remarks: While the bound $\zeta(s) \ll |t|$ on the critical strip (away from $s = 1$) is more than sufficient for our present purpose, we can do considerably better. To begin with, we can choose some N and leave alone the terms with $n < N$ in $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, using the integral approximation only for the $n \geq N$ terms:

$$\zeta(s) = \sum_{n=1}^{N-1} n^{-s} + \frac{N^{1-s}}{s-1} + \sum_{n=N}^{\infty} \int_n^{n+1} (n^{-s} - x^{-s})\,dx.$$

For large t, N this is $\ll N^{1-\sigma} + |t|N^{-\sigma}$, uniformly at least for $\sigma \geq 1/2$. Taking $N = |t| + O(1)$ we find $\zeta(\sigma+it) \ll |t|^{1-\sigma}$ for $\sigma \geq 1/2$, $|t| > 1$. At the end of this chapter we describe further improvements. Meanwhile, note our choice of $N = |t| + O(1)$, which may seem suboptimal. We wanted to make the bound as good as possible, that is, to minimize $N^{1-\sigma} + |t|N^{-\sigma}$. In calculus we learned to minimize such an expression by setting its derivative equal to zero. That would give N proportional to $|t|$, but we arbitrarily set the constant of proportionality to 1 even though another choice would make $N^{1-\sigma} + |t|N^{-\sigma}$ somewhat smaller. In general when we bound some quantity by a sum $O(f(N) + g(N))$ of an increasing and a decreasing function of some parameter N, we shall simply choose N so that $f(N) = g(N)$ (or, if N is constrained to be an integer, so that

$f(N)$ and $g(N)$ are nearly equal). This is much simpler and less error-prone than fumbling with derivatives, and is sure to give the minimum to within a factor of 2, which is good enough when we are dealing with $O(\cdots)$ bounds.

Product and logarithmic-derivative formulas. By our general product formula for an entire function of finite order we know that $\xi(s)$ has a product expansion:

$$\xi(s) = \frac{e^{A+Bs}}{s^2 - s} \prod_\rho (1 - (s/\rho)) e^{s/\rho}, \tag{1}$$

for some constants A, B, with the product ranging over zeros ρ of ξ (that is, the nontrivial zeros of ζ) listed with multiplicity. Moreover, $\sum_\rho |\rho|^{-1-\epsilon} < \infty$ for all $\epsilon > 0$, but $\sum_\rho |\rho|^{-1} = \infty$ because $\xi(s)$ is not $O(\exp C'|s|)$. The logarithmic derivative of (1) is

$$\frac{\xi'}{\xi}(s) = B - \frac{1}{s} - \frac{1}{s-1} + \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right); \tag{2}$$

since $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ we also get a product formula for $\zeta(s)$, and a partial-fraction expansion of its logarithmic derivative:

$$\frac{\zeta'}{\zeta}(s) = B - \frac{1}{s-1} + \frac{1}{2}\log\pi - \frac{1}{2}\frac{\Gamma'}{\Gamma}(\frac{s}{2}+1) + \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right). \tag{3}$$

(We have shifted from $\Gamma(s/2)$ to $\Gamma(s/2+1)$ to absorb the term $-1/s$; note that $\zeta(s)$ does not have a pole or zero at $s = 0$.)

Vertical distribution of zeros. Since the zeros ρ of $\xi(s)$ are limited to a strip we can find much more precise information about the distribution of their sizes than the convergence of $\sum_\rho |\rho|^{-1-\epsilon}$ and the divergence of $\sum_\rho |\rho|^{-1}$. Let $N(T)$ be the number of zeros in the rectangle $\sigma \in [0,1]$, $t \in [0,T]$; note that this is very nearly half of what we would call $n(T)$ in the context of the general product formula for $(s^2 - s)\xi(s)$.

Theorem (von Mangoldt)**.** *As* $T \to \infty$*,*

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T). \tag{4}$$

Proof: We follow chapter 15 of [Davenport 1967], keeping in mind that Davenport's ξ and ours differ by a factor of $(s^2 - s)/2$.

We may assume that T does not equal the imaginary part of any zero of $\zeta(s)$. Then

$$2N(T) - 2 = \frac{1}{2\pi i}\oint_{C_R} \frac{\xi'}{\xi}(s)\, ds = \frac{1}{2\pi i}\oint_{C_R} d(\log\xi(s)) = \frac{1}{2\pi}\oint_{C_R} d(\operatorname{Im}\log\xi(s)),$$

where C_R is the boundary of the rectangle $\sigma \in [-1,2]$, $t \in [-T,T]$. Since $\xi(s) = \xi(1-s) = \overline{\xi(\bar{s})}$, we may by symmetry evaluate the last integral by integrating over a quarter of C_R and multiplying by 4. We use the top right quarter, going from 2 to $2+iT$ to $(1/2)+iT$. Because $\log\xi(s)$ is real at $s=2$, we have

$$\pi(N(T)-1) = \operatorname{Im}\log\xi(\frac{1}{2}+iT) = \operatorname{Im}(\log\Gamma(\frac{1}{4}+\frac{iT}{2})) - \frac{T}{2}\log\pi + \operatorname{Im}(\log\zeta(\frac{1}{2}+iT)).$$

By Stirling, the first term is within $O(T^{-1})$ of

$$\operatorname{Im}\left((\frac{iT}{2}-\frac{1}{4})\log(\frac{iT}{2}+\frac{1}{4})\right) - \frac{T}{2}$$

$$= \frac{T}{2}\log\left|\frac{iT}{2}+\frac{1}{4}\right| - \frac{1}{4}\operatorname{Im}\log\big(\frac{iT}{2}+\frac{1}{4}\big) - \frac{T}{2} = \frac{T}{2}\big(\log\frac{T}{2}-1\big) + O(1).$$

Thus (4) is equivalent to

$$\operatorname{Im}\log\zeta(\frac{1}{2}+iT) \ll \log T. \tag{5}$$

We shall show that for $s=\sigma+it$ with $\sigma\in[-1,2]$, $|t|>1$ we have

$$\frac{\zeta'}{\zeta}(s) = \sum_{|\operatorname{Im}(s-\rho)|<1} \frac{1}{s-\rho} + O(\log|t|), \tag{6}$$

and that the sum comprises at most $O(\log|t|)$ terms, from which our desired estimate will follow by integrating from $s=2+iT$ to $s=1/2+iT$. We start by taking $s=2+it$ in (3). At that point the LHS[1] is uniformly bounded (use the Euler product), and the RHS is

$$\sum_\rho\left(\frac{1}{2+it-\rho}+\frac{1}{\rho}\right) + O(\log|t|)$$

by Stirling. Thus the sum, and in particular its real part, is $O(\log|t|)$. But each summand has positive real part, which is at least $1/(4+(t-\operatorname{Im}\rho)^2)$. Our second claim, that $|t-\operatorname{Im}\rho|<1$ holds for at most $O(\log|t|)$ zeros ρ, follows immediately. It also follows that

$$\sum_{|\operatorname{Im}(s-\rho)|\geq 1} \frac{1}{\operatorname{Im}(s-\rho)^2} \ll \log|t|.$$

Now by (3) we have

$$\frac{\zeta'}{\zeta}(s) - \frac{\zeta'}{\zeta}(2+it) = \sum_\rho\Big(\frac{1}{s-\rho} - \frac{1}{2+it-\rho}\Big) + O(1).$$

[1] "LHS" and "RHS" are the left-hand side and right-hand side of an equation.

The LHS differs from that of (6) by $O(1)$, as noted already; the RHS summed over zeros with $|\mathrm{Im}(s-\rho)| < 1$ is within $O(\log|t|)$ of the RHS of (6); and the remaining terms are

$$(2-\sigma) \sum_{|\mathrm{Im}(s-\rho)|\geq 1} \frac{1}{(s-\rho)(2+it-\rho)} \ll \sum_{|\mathrm{Im}(s-\rho)|\geq 1} \frac{1}{\mathrm{Im}(s-\rho)^2} \ll \log|t|.$$

This proves (6) and thus also (5); von Mangoldt's theorem (4) follows. $\square$

For much more about the vertical distribution of the nontrivial zeros ρ of $\zeta(s)$ see [Titchmarsh 1951], Chapter 9.

Further remarks

Recall that as part of our proof that $(s^2-s)\xi(s)$ has order 1 we showed that for each σ there exists $\nu(\sigma)$ such that $|\zeta(\sigma+it)| \ll |t|^{\nu(\sigma)}$ as $|t|\to\infty$. Any bounded $\nu(\sigma)$ suffices for this purpose, but one naturally asks how small $\nu(\sigma)$ can become. Let $\mu(\sigma)$ be the infimum of all such $\nu(\sigma)$; that is,

$$\mu(\sigma) := \limsup_{|t|\to\infty} \frac{\log|\zeta(\sigma+it)|}{\log|t|}.$$

We have seen that $\mu(\sigma) = 0$ for $\sigma > 1$, that $\mu(1-\sigma) = \mu(\sigma) + \sigma - \frac{1}{2}$ by the functional equation (so in particular $\mu(\sigma) = \frac{1}{2} - \sigma$ for $\sigma < 0$), and that $\mu(\sigma) \leq 1$ for $\sigma < 1$, later improving the upper bound from 1 to $1-\sigma$. For $\sigma \in (0,1)$ an even better bound is obtained using the "approximate functional equation" for $\zeta(s)$ (usually attributed to Siegel, but now known to have been used by Riemann himself) to show that $\mu(\sigma) \leq (1-\sigma)/2$; this result and the fact that $\mu(\sigma) \geq 0$ for all σ, also follow from general results in complex analysis, which indicate that since $\mu(\sigma) < \infty$ for all σ the function $\mu(\cdot)$ must be convex. For example, $\mu(1/2) \leq 1/4$, so $|\zeta(\frac{1}{2}+it)| \ll_\epsilon |t|^{\frac{1}{4}+\epsilon}$.

The value of $\mu(\sigma)$ is not known for any $\sigma \in (0,1)$. The *Lindelöf conjecture* asserts that $\mu(1/2) = 0$, from which it would follow that $\mu(\sigma) = 0$ for all $\sigma \geq 1/2$ while $\mu(\sigma) = \frac{1}{2} - \sigma$ for all $\sigma \leq 1/2$. Equivalently, the Lindelöf conjecture asserts that $\zeta(\sigma+it) \ll_\epsilon |t|^\epsilon$ for all $\sigma \geq 1/2$ (excluding a neighborhood of the pole $s = 1$), and thus by the functional equation that also $\zeta(\sigma+it) \ll_\epsilon |t|^{1/2-\sigma+\epsilon}$ for all $\sigma \leq 1/2$. We shall see that this conjecture is implied by the Riemann hypothesis, and also that it holds on average in the sense that $\int_0^T |\zeta(\frac{1}{2}+it)|^2\,dt \ll T^{1+\epsilon}$. However, the best upper bound currently proved on $\mu(1/2)$ is only a bit smaller than 1/6; when we get to exponential sums later this term we shall derive the upper bound of 1/6.

Exercises

1. Show that in the product formula (1) we may take $A = 0$. Prove the formula

$$\boldsymbol{\gamma} = \lim_{s\to 1}\Big(\zeta(s) - \frac{1}{s-1}\Big)$$

for Euler's constant, and use it to compute

$$B = \lim_{s\to 0}\Big(\frac{\xi'}{\xi}(s) + \frac{1-s}{s}\Big) = \lim_{s\to 1}\Big(-\frac{\xi'}{\xi}(s) + \frac{s}{1-s}\Big)$$
$$= \frac{1}{2}\log 4\pi - 1 - \frac{\gamma}{2} = -0.0230957\ldots.$$

Show also (starting by pairing the ρ and $\bar{\rho}$ terms in the infinite product) that

$$B = -\sum_{\rho} \mathrm{Re}(\rho)/|\rho|^2,$$

and thus that $|\mathrm{Im}(\rho)| > 6$ for every nontrivial zero ρ of $\zeta(s)$. [From [Davenport 1967], Chapter 12. It is known that in fact the smallest zeros have (real part 1/2 and) imaginary part $\pm 14.134725\ldots$]

2. By the functional equation, $\xi(s)$ can be regarded as an entire function of $s^2 - s$; what is the order of this function? Use this to obtain the alternative infinite product

$$\xi(s) = \frac{\xi(1/2)}{4(s-s^2)} \prod_{\rho}^{+} \left[1 - \left(\frac{s-1/2}{\rho-1/2}\right)^2\right], \tag{7}$$

the product extending over zeros ρ of ξ whose imaginary part is positive. [This symmetrical form eliminates the exponential factors e^{A+Bs} and $e^{s/\rho}$ of (1). We nevertheless use (1) rather than (7) to develop the properties of ξ and ζ, because (1) generalizes to arbitrary Dirichlet L-series, whereas (7) generalizes only to Dirichlet series associated to real characters χ since in general the functional equation relates $L(s,\chi)$ with $L(1-s,\overline{\chi})$, not with $L(1-s,\chi)$.]

3. Let f be any analytic function on the vertical strip $a < \sigma < b$ such that

$$M_f(\sigma) := \limsup_{|t|\to\infty} \frac{\log|f(\sigma+it)|}{\log|t|}$$

is finite for all $\sigma \in (a,b)$. Prove that M_f is a convex function on that interval. [Hint: Apply the maximum principle to αf for suitable analytic functions $\alpha(s)$.]

It follows in particular that M_f is continuous on (a,b). While $\zeta(s)$ is not analytic on vertical strips that contain $s = 1$, we can still deduce the convexity of $\mu : \mathbf{R}\to\mathbf{R}$ from $\mu(\sigma) = M_f(\sigma)$ for $f(s) = \zeta(s) - (1/(s-1))$.

Much the same argument proves the "three lines theorem": if f is actually bounded on the strip then $\log\sup_t |f(\sigma+it)|$ is a convex function of σ. The name of this theorem alludes to the equivalent formulation: if $a < \sigma_1 < \sigma_2 < \sigma_3 < b$ then the supremum of $|f(s)|$ on the line $s = \sigma_2 + it$ is bounded by a weighted geometric mean of its suprema on the lines $s = \sigma_1 + it$ and $s = \sigma_3 + it$.

Reference

[Titchmarsh 1951] Titchmarsh, E.C.: *The Theory of the Riemann Zeta-Function*. Oxford: Clarendon, 1951. [HA 9.51.14 / QA351.T49; 2nd ed. revised by D.R. Heath-Brown 1986, QA246.T44]