Math 229: Introduction to Analytic Number Theory

The asymptotic formula for primes in arithmetic progressions;
the Extended Riemann Hypothesis, concerning the zeros of $L(s, \chi)$

Now that we have the functional equation for $L(s, \chi)$, we can recover the asymptotics for $\psi(x, \chi)$, and thus also for $\psi(x, a \bmod q)$ and $\pi(x, a \bmod q)$, just as we did for $\psi(x)$ and $\pi(x)$ — at least if we are not very concerned with how the implied constants depend on q. We also state the Extended Riemann Hypothesis and relate it with conjectural improvements of the error estimates. The proofs are similar enough that we relegate most of the details to the Exercises. We shall soon see how to better control the dependence of our estimates on q. But the relatively crude bounds below are still of interest because these bounds, but not the later improvements for $L(s, \chi)$, generalize to other Dirichlet series such as the zeta functions of number fields; see the final Exercise.

Let χ be a primitive character mod $q > 1$. We readily adapt our argument showing that $(s^2 - s)\xi(s)$ is an entire function of order 1 to show that $\xi(s, \chi)$ is an entire function of order 1, and thus has a Hadamard product

$$\xi(s, \chi) = \xi(0, \chi)e^{Bs}\prod_{\rho}(1 - s/\rho)e^{s/\rho}. \tag{1}$$

The product ranges over zeros of $\xi(s, \chi)$, counted with multiplicity; that is, ρ ranges over the "nontrivial zeros" of $L(s, \chi)$, those with $\sigma \in [0, 1]$ except for the simple zero at $s = 0$ when $\chi(1) = +1$. Thus

$$\frac{\xi'}{\xi}(s, \chi) = B + \sum_{\rho}\left(\frac{1}{s - \rho} + \frac{1}{\rho}\right). \tag{2}$$

Note that B depends on χ; see Exercise 1 below. Fortunately it will usually cancel out from our formulas. It follows that

$$\frac{L'}{L}(s, \chi) = B - \frac{1}{2}\log\frac{q}{\pi} - \frac{1}{2}\frac{\Gamma'}{\Gamma}((s + \mathsf{a})/2) + \sum_{\rho}\left(\frac{1}{s - \rho} + \frac{1}{\rho}\right). \tag{3}$$

How are these zeros ρ distributed? We noted already that their real parts lie in $[0, 1]$. If $L(\rho, \chi) = 0$ then by the functional equation $0 = L(1 - \rho, \overline{\chi}) = L(1 - \bar{\rho}, \chi)$. Thus the zeros are symmetrical about the line $\sigma = 1/2$, but not (unless χ is real) about the real axis. So the proper analog of $N(T)$ is half of $N(T, \chi)$, where $N(T, \chi)$ is defined as the number of zeros of $L(s, \chi)$ in $\sigma \in (0, 1)$, $|t| < T$, counted with multiplicity. [NB this excludes the trivial zero at $s = 0$, which occurs for even χ.] Again we evaluate this by integrating ξ'/ξ around a rectangle. The new factor $q^{s/2}$ in $\xi(s, \chi)$ introduces an extra term of $(T/2\pi)\log q$ into the formula for $N(T, \chi)/2$. That factor is also responsible for the new term $-\frac{1}{2}\log q$ in (3), which forces us to subtract $O(\log q)$ from our lower bound on the real part of

$(L'/L)(s, \chi)$. This bound now becomes

$$\frac{L'}{L}(s, \chi) = \sum_{|\mathrm{Im}(s-\rho)|<1} \frac{1}{s-\rho} + O(\log |qt|)$$

$(\sigma \in [-1, 2])$, the sum comprising $O(\log |qt|)$ terms. We conclude:

Theorem. *The estimate*

$$\frac{1}{2} N(T, \chi) = \frac{T}{2\pi} \log \frac{qT}{2\pi} - \frac{T}{2\pi} + O(\log qT) \tag{4}$$

holds for all $T \geq 2$, with an implied constant independent of q.

(The lower bound on T could be replaced by any $T_0 > 1$, possibly changing the implied constant if $q = 1$ and T_0 is too close to 1.)

Proof: See Exercise 2. $\square$

To isolate the primes in arithmetic progressions mod q, we need also characters that are not primitive, such as χ_0. Let χ_1 be the primitive character mod $q_1 | q$ underlying a nonprimitive χ mod q. Then

$$L(s, \chi) = \prod_{p|q} \left(1 - \chi_1(p)p^{-s}\right) \cdot L(s, \chi_1).$$

The elementary factor $\prod_{p|q}(1 - \chi_1(p)p^{-s})$ has, for each p dividing q but not q_1, a total of $(T/\pi) \log p + O(1)$ purely imaginary zeros of absolute value $< T$. This, together with the estimate (4) for $N(T, \chi_1)$, shows that the RHS of (4) is an upper bound on $\frac{1}{2} N(T, \chi)$, even when χ is not primitive.

The horizontal distribution of ρ is subtler. We noted already that the logarithmic derivative of

$$\zeta_q(s) := \prod_{\chi \bmod q} L(s, \chi)$$

is a Dirichlet series $-\sum_n \Lambda_q(n) n^{-s}$ with $\Lambda_q(n) \geq 0$ for all n, and deduced that the $3 + 4\cos\theta + \cos 2\theta$ trick shows that ζ_q, and thus each factors $L(s, \chi)$, does not vanish at $s = 1 + it$. We can then adapt the proof of the classical zero-free region for $\zeta(s)$; since, however, $\zeta_q(s)$ is the product of $\varphi(q)$ L-series, each of which contributes $O(\log |qt|)$ to the bound on $(\zeta_q'/\zeta_q)(\sigma + it)$, the resulting zero-free region is not $1 - \sigma < c / \log |t|$ or even $1 - \sigma < c / \log |qt|$ but

$$1 - \sigma < \frac{c}{(\varphi(q) \log |qt|)}.$$

Moreover, the fact that this holds only for say $|t| > 2$ is newly pertinent: unlike $\zeta(s)$, the L-series might have zeros of small imaginary part. [Indeed it is known that there are Dirichlet L-series that vanish on points arbitrarily close to the real axis.] Still, for every q there are only finitely many zeros with $|t| \leq 2$. So our formula

$$\psi(x, \chi) = \frac{1}{2\pi i} \int_{1+\frac{1}{\log x}-iT}^{1+\frac{1}{\log x}+iT} -\frac{L'}{L}(s, \chi) x^s \frac{ds}{s} + O\left(\frac{x \log^2 x}{T}\right) \qquad (T \in [1, x])$$

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yields an estimate as before, with only the difference that when $\chi \neq \chi_0$ there is no "main term" coming from a pole at $s = 1$. We thus find

$$\psi(x, \chi) \ll_\chi x \exp(-C_\chi \sqrt{\log x}) \tag{5}$$

for some constant $C_\chi > 0$. Multiplying by $\overline{\chi}(a)$ and averaging over χ (including χ_0, for which $\psi(x, \chi_0) = x + O(x \exp(-C\sqrt{\log x}))$ instead of (5), we obtain

$$\psi(x, a \bmod q) = \frac{1}{\varphi(q)} x + O_q(x \exp(-C_q \sqrt{\log x})), \tag{6}$$

and thus

$$\pi(x, a \bmod q) = \frac{1}{\varphi(q)} \operatorname{li}(x) + O_q(x \exp(-C_q \sqrt{\log x})). \tag{7}$$

Note however that the dependence of the error terms on q is unpredictable. The zero-free region depends explicitly on q (though as we shall see it need not shrink nearly as fast as $1/(\varphi(q) \log q)$, a factor which alone would make C_q proportional to $(\varphi(q) \log q)^{-1/2}$), but it excludes a neighborhood of the real axis. It would then seem that to specify C_χ and $\ll_\chi$ we would have to compute for each χ the largest $\operatorname{Re}(\rho)$. There's also the matter of the contribution of the B's from (3).

Consider, by comparison, the consequences of the *Extended Riemann Hypothesis* (ERH), which is the conjecture that each nontrivial zero ρ of an L-series associated to a primitive Dirichlet character χ has real part $1/2$.[1] Our analysis of $\psi(x)$ under RH then carries over almost verbatim to show that $\psi(x, \chi) \ll x^{1/2} \log^2 x$ with an absolute and effective implied constant as long as $q < x$, and thus that

$$\psi(x, a \bmod q) = \frac{x}{\varphi(q)} + O(x^{1/2} \log^2 x) \tag{8}$$

holds if $L(s, \chi)$ satisfies ERH for all Dirichlet characters mod q_1 with $q_1 | q$, again with the O-constant effective and independent of q. It would also follow that

$$\pi(x, a \bmod q) = \frac{\operatorname{li}(x)}{\varphi(q)} + O(x^{1/2} \log x). \tag{9}$$

Again there are some remarkable effects of the difference between $\psi(x, a \bmod q)$ and the sum of $\log p$ over the primes counted in $\pi(x, a \bmod q)$. Most strikingly, $\pi(x, \chi)$ tends to be negative for nontrivial real characters χ, because the contribution of $\sum_{n<x} \chi(n^2) \Lambda(n^2)$ to $\psi(x, \chi)$ is asymptotic to $+\sqrt{x}$. For instance, $\pi(x, 1 \bmod 4) < \pi(x, 3 \bmod 4)$ for "most" x, where one might intuitively expect that $\pi(x, \chi_4)$ could as easily be positive as negative. This is the "Chebyshev's Bias" of the title of [RS 1994] and [BFHR 2001], to which we refer for more precise statements, as well as subtler effects of this kind and numerical computations of the theoretical and "experimental" sizes of these biases.

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[1] Attributed by Davenport to "Piltz in 1884" (page 129). We distinguish ERH from the *Generalized Riemann Hypothesis* (GRH), which pertains to much more general Dirichlet series, such as a zeta function of a number field or the L-series attached to a modular form.

Exercises

1. Show that the real part of the term B of (2) is $-\sum_\rho \mathrm{Re}(1/\rho)$. Conclude that $\mathrm{Re}(B) < 0$. [Davenport 1967, page 85.]

In the next three Exercises you will fill in the proofs outlined above of formulas for $N(T, \chi)$, $\psi(x, \chi)$ $\psi(x, a \bmod q)$, and $\pi(x, a \bmod q)$. Again the complete proofs may be found in [Davenport 1967] and elsewhere.

2. Complete the missing steps in the proof of (4).

3. Complete the missing steps in the proof of (5), (6), and (7).

4. Verify that, under the relevant ERH, the O-constant in (9) does not depend on q. Obtain an analogous estimate on the weaker assumption that ζ_q has no zeros of real part $> \theta$ for some $\theta \in (\frac{1}{2}, 1)$. Show that if for some q we have $\pi(x, a \bmod q) \ll_\epsilon x^{\theta+\epsilon}$ for all $a \in (\mathbf{Z}/q)^*$ then all the $L(s, \chi)$ for Dirichlet characters $\chi \bmod q$ are nonzero on $\sigma > \theta$.

5. [The Prime Number Theorem for number fields] Let K be a number field of degree $n = r_1 + 2r_2$. We already defined the zeta function

$$\zeta_K(s) := \sum_I |I|^{-s} = \prod_\wp (1 - |\wp|^{-s})^{-1} \qquad (\sigma > 1),$$

in which $|I|$ is the norm of an ideal I, and the sum and product extend respectively over nonzero ideals I and prime ideals $\wp$ of the ring of integers O_K. We also reported that ζ_K is known to extend to a meromorphic function on $\mathbf{C}$, regular except for a simple pole at $s = 1$, that satisfies a functional equation $\xi_K(s) = \xi_K(1 - s)$, where

$$\xi_K(s) := \Gamma(s/2)^{r_1} \Gamma(s)^{r_2} (4^{-r_2} \pi^{-n} |d|)^{s/2} \zeta_K(s)$$

and d is the discriminant of K. In particular, $(s^2 - s)\xi_K(s)$ is an entire function; it is also known that it is an entire function of order 1. Use this to obtain an approximation of the number of nontrivial zeros ρ of ζ_K such that $|\mathrm{Im}(\rho)| \leq T$, with an error estimate depending explicitly on n and $|d|$. Obtain a zero-free region for ζ_K, and deduce that

$$\#\{\wp : |\wp| \leq x\} = \mathrm{li}(x) + O(x \exp(-C_K \sqrt{\log x}))$$

for some constant $C_K > 0$.