Math 229: Introduction to Analytic Number Theory

A nearly zero-free region for $L(s,\chi)$, and Siegel's theorem

We used positivity of the logarithmic derivative of ζ_q to get a crude zero-free region for $L(s,\chi)$. Better zero-free regions can be obtained with some more effort by working with the $L(s,\chi)$ individually. The situation is most satisfactory for complex χ, that is, for characters with $\chi^2 \neq \chi_0$. (Recall that real χ were also the characters that gave us the most difficulty in the proof of $L(1,\chi) \neq 0$; it is again in the neighborhood of $s = 1$ that it is hard to find a good zero-free region for the L-function of a real character.)

To obtain the zero-free region for $\zeta(s)$, we started with the expansion of the logarithmic derivative

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \Lambda(n) n^{-s} \qquad (\sigma > 1)$$

and applied the inequality

$$0 \leq \frac{1}{2}(z + 2 + \bar{z})^2 = \mathrm{Re}(z^2 + 4z + 3) = 3 + 4\cos\theta + \cos 2\theta \qquad (z = e^{i\theta})$$

to the phases $z = n^{-it} = e^{i\theta}$ of the terms n^{-s}. To apply the same inequality to

$$-\frac{L'}{L}(s,\chi) = \sum_{n=1}^{\infty} \chi(n)\Lambda(n) n^{-s},$$

we must use $z = \chi(n) n^{-it}$ instead of n^{-it}, obtaining

$$0 \leq \mathrm{Re}\big(3 + 4\chi(n) n^{-it} + \chi^2(n) n^{-2it}\big).$$

Multiplying by $\Lambda(n) n^{-\sigma}$ and summing over n yields

$$0 \leq 3\left[-\frac{L'}{L}(\sigma,\chi_0)\right] + 4\,\mathrm{Re}\left[-\frac{L'}{L}(\sigma+it,\chi)\right] + \mathrm{Re}\left[-\frac{L'}{L}(\sigma+2it,\chi^2)\right]. \qquad (1)$$

Now we see why the case $\chi^2 = \chi_0$ will give us trouble near $s = 1$: for such χ the last term in (1) is within $O(1)$ of $\mathrm{Re}(-(\zeta'/\zeta)(\sigma+2it))$, so the pole of $(\zeta'/\zeta)(s)$ at $s = 1$ will undo us for small $|t|$.

Let us see how far (1) does take us. Our bounds will involve $\log q$ for small $|t|$, and $\log q|t|$ for large $|t|$. To cover both ranges, and also to accommodate the case $q = 1$ (the Riemann zeta function), we use the convenient and conventional abbreviation

$$\mathscr{L} := \log q(|t|+2).$$

We shall prove:

Theorem. *There is a constant $c > 0$ such that if $L(\sigma + it, \chi) = 0$ for some primitive complex Dirichlet character χ mod q then*

$$\sigma < 1 - \frac{c}{\mathscr{L}}\,. \tag{2}$$

If χ is a real primitive character then (2) holds for all zeros of $L(s, \chi)$ with at most one exception. The exceptional zero, if it exists, is real and simple.

Proof: We again apply (1) for suitable σ, t with $1 < \sigma \leq 2$. The first term is

$$\leq 3\left[-\frac{\zeta'}{\zeta}(\sigma)\right] < \frac{3}{\sigma - 1} + O(1).$$

For the remaining terms, we use the partial-fraction expansion

$$-\frac{L'}{L}(s, \chi) = \frac{1}{2}\log\frac{q}{\pi} + \frac{1}{2}\frac{\Gamma'}{\Gamma}((s + \mathsf{a})/2) - B_\chi - \sum_\rho \left(\frac{1}{s - \rho} + \frac{1}{\rho}\right).$$

To eliminate the contributions of B_χ and $\sum_\rho 1/\rho$ we use this formula to evaluate $(L'/L)(2, \chi) - (L'/L)(s, \chi)$. By the Euler product we have $(L'/L)(2, \chi) = O(1)$. This leaves

$$-\frac{L'}{L}(s, \chi) = \frac{1}{2}\frac{\Gamma'}{\Gamma}((s + \mathsf{a})/2) - \sum_\rho \left(\frac{1}{s - \rho} - \frac{1}{2 - \rho}\right) + O(1).$$

Next take real parts. For ρ of real part in $[0, 1]$ we have $\mathrm{Re}(1/(2-\rho)) \ll |2-\rho|^{-2}$. To estimate the sum of this over all ρ, we may apply Jensen's theorem to $\xi(2+s, \chi)$, finding that the number (with multiplicity) of $|\rho|$ at distance at most r from 2 is $O(r \log qr)$, and thus by partial summation that $\sum_\rho |2 - \rho|^{-2} \ll \log q$. We estimate the real part of the Γ'/Γ term by Stirling as usual, and find

$$\mathrm{Re}\left[-\frac{L'}{L}(s, \chi)\right] < O(\mathscr{L}) - \sum_\rho \mathrm{Re}\,\frac{1}{s - \rho}.$$

Again each of the $\mathrm{Re}(1/(s - \rho))$ is nonnegative, so the estimate remains true if we include only some of the zeros ρ, or none of them.

In particular it follows that

$$\mathrm{Re}\left[-\frac{L'}{L}(\sigma + 2it, \chi^2)\right] < O(\mathscr{L}), \tag{3}$$

at least when χ^2 is a primitive character. If χ^2 is not primitive, but still not the trivial character χ_0, then (3) holds when χ^2 is replaced by its corresponding primitive character; but the error thus introduced is at most

$$\sum_{p|q} \frac{p^{-\sigma}}{1 - p^{-\sigma}} \log p < \sum_{p|q} \log p \leq \log q < \mathscr{L},$$

so can be absorbed into the $O(\mathscr{L})$ error. But when $\chi^2 = \chi_0$ the partial-fraction expansion of its $-L'/L$ has a term $+1/(s-1)$ which cannot be discarded, and can absorbed into $O(\mathscr{L})$ only if s is far enough from 1. We thus conclude that (3) holds unless $\chi^2 = \chi_0$ and $|t| < c/\log q$, the implied constant in (3) depending on c. (Equivalently, we could change $O(\mathscr{L})$ to $1/|t| + O(\mathscr{L})$ when $\chi^2 = \chi_0$.)

The endgame is the same as we have seen for the classical zero-free region for $\zeta(s)$: if there is a zero $\rho = 1 - \delta + it$ with δ small, use its imaginary part t in (1) and find from the partial-fraction expansion that

$$\operatorname{Re}\left[-\frac{L'}{L}(\sigma + it, \chi)\right] < O(\mathscr{L}) - \frac{1}{\sigma - \operatorname{Re}(\rho)}.$$

Combining this with our previous estimates yields

$$\frac{4}{\sigma + \delta - 1} < \frac{3}{\sigma - 1} + O(\mathscr{L});$$

choosing $\sigma = 1 + 4\delta$ as before yields $1/\delta \ll O(\mathscr{L})$ under the hypotheses of (3), completing the proof of (2) with the possible exception of real χ and zeros of imaginary part $\ll 1/\log q$.

Next suppose that χ is a real character and fix some $\delta > 0$, to be chosen later. We have a zero-free region for $|t| \geq \delta/\log q$. To deal with zeros of small imaginary part, let $s = \sigma$ in (1) — or, more simply, use the inequality[1] $1 + \operatorname{Re}(e^{i\theta}) \geq 0$ — to find

$$\sum_{|\operatorname{Im}(\rho)| < \delta/\log q} \operatorname{Re}(1/(\sigma - \rho)) < \frac{1}{\sigma - 1} + O(\log q),$$

the implied O-constant not depending on δ. Each term $\operatorname{Re}(1/(\sigma - \rho))$ equals $\operatorname{Re}(\sigma - \rho)/|\sigma - \rho|^2$. We choose $\sigma = 1 + (2\delta/\log q)$. Then $|\operatorname{Im}(\rho)| < \frac{1}{2}(\sigma - 1) < \frac{1}{2}\operatorname{Re}(\sigma - \rho)$, and thus $|\sigma - \rho|^2 < \frac{5}{4}\operatorname{Re}(\sigma - \rho)^2$ and $\operatorname{Re}(1/(\sigma - \rho)) > \frac{4}{5}/\operatorname{Re}(\sigma - \rho)$. So,

$$\frac{4}{5} \sum_{|\operatorname{Im}(\rho)| < \delta/\log q} \left(1 - \operatorname{Re}(\rho) + \frac{2\delta}{\log q}\right)^{-1} < \frac{\log q}{2\delta} + A \log q,$$

for some constant A independent of the choice of δ. Therefore, if δ is small enough, we can find $c > 0$ such that at most one ρ can have real part greater than $1 - (c/\log q)$. (Specifically, we may choose any $\delta < 3/10A$, and take $c = 2\delta(3 - 10A\delta)/5(1 + 2A\delta)$.) Since ρ's are counted with multiplicity and come in complex conjugate pairs, it follows that this exceptional zero, if it exists, is real and simple. □

This exceptional zero is usually denoted by β. Of course we expect, by the Extended Riemann Hypothesis, that there is no such β. The nonexistence of β, though much weaker than ERH, has yet to be proved; but we can still obtain some strong restrictions on how β can vary with q and χ. We begin by showing

[1] That is, use the positivity of $-\zeta'_\chi/\zeta_\chi$, where $\zeta_\chi(s) = \zeta(s)L(s,\chi)$ is the zeta function of the quadratic number field corresponding to χ.

that at most one of the Dirichlet characters mod q can have an L-series with an exceptional zero, and deduce a stronger estimate on the error terms in our approximate formulas for $\psi(x, a \bmod q)$ and $\pi(x, a \bmod q)$. Since χ need not be primitive, it follows that in fact β cannot occur even for characters of different moduli if we set the threshold low enough:

Theorem. [Landau 1918] *There is a constant $c > 0$ such that, for any distinct primitive real characters χ_1, χ_2 to (not necessarily distinct) moduli q_1, q_2 at most one of $L(s, \chi_1)$ and $L(s, \chi_2)$ has an exceptional zero $\beta > 1 - c/\log q_1 q_2$.*

Proof: Since χ_1, χ_2 are distinct primitive real characters, their product $\chi_1\chi_2$, while not necessarily primitive, is also a nontrivial Dirichlet character, with modulus at most $q_1 q_2$. Hence $-(L'/L)(\sigma, \chi_1\chi_2) < O(\log q_1 q_2)$ for $\sigma > 1$. Let[2]

$$F(s) = \zeta(s)L(s, \chi_1)L(s, \chi_2)L(s, \chi_1\chi_2) \tag{4}$$

Then $-F'/F$ is the sum of the negative logarithmic derivatives of $\zeta(s)$, $L(s, \chi_1)$, $L(s, \chi_2)$, and $L(s, \chi_1\chi_2)$, which is the positive Dirichlet series

$$\sum_{n=1}^{\infty}(1 + \chi_1(n))(1 + \chi_2(n))\Lambda(n)n^{-s}.$$

In particular, this series is positive for real $s > 1$. Arguing as before, we find that if β_i are exceptional zeros of $L(s, \chi_i)$ then

$$\frac{1}{\sigma - \beta_1} + \frac{1}{\sigma - \beta_2} < \frac{1}{\sigma - 1} + O(\log q_1 q_2);$$

if $\beta_i > 1 - \delta$ then we may take $\sigma = 1 + 2\delta$ to find $1/6\delta < O(\log q_1 q_2)$, whence $\delta \gg 1/\log q_1 q_2$ as claimed. $\Box$

In particular, for each q there is at most one real character mod q whose L-series has an exceptional zero $\beta > 1 - (c/\log q)$. This lets us obtain error terms that depend explicitly on q and (if it exists) β in the asymptotic formulas for $\psi(x, a \bmod q)$ and $\pi(x, a \bmod q)$. For instance (see e.g. Chapter 20 of [Davenport 1967]), we have:

Theorem. *For every $C > 0$ there exists $c > 0$ such that whenever* $\gcd(a, q) = 1$ *we have*

$$\psi(x, a \bmod q) = \left(1 + O(\exp -c\sqrt{\log x})\right)\frac{x}{\varphi(q)} \tag{5}$$

and

$$\pi(x, a \bmod q) = \left(1 + O(\exp -c\sqrt{\log x})\right)\frac{\mathrm{li}(x)}{\varphi(q)} \tag{6}$$

[2] If $\chi_1\chi_2$ is itself primitive then $F(s)$ is the zeta function of a biquadratic number field K, namely the compositum of the quadratic fields corresponding to χ_1 and χ_2. In general $\zeta_K(s) = \zeta(s)L(s, \chi_1)L(s, \chi_2)L(s, \chi_3)$ where χ_3 is the primitive character underlying $\chi_1\chi_2$; thus $F(s)$ always equals $\zeta_K(s)$ multiplied by a finite Euler product.

for all $x > \exp(C \log^2 q)$*, unless there is a Dirichlet character* $\chi \bmod q$ *for which* $L(s, \chi)$ *has an exceptional zero* β*, in which case*

$$\psi(x, a \bmod q) = \left(1 - \frac{\chi(a) x^{\beta-1}}{\beta} + O(\exp -c\sqrt{\log x})\right) \frac{x}{\varphi(q)} \tag{7}$$

and

$$\pi(x, a \bmod q) = \frac{1}{\varphi(q)} \left(\mathrm{li}(x) - \chi(a)\,\mathrm{li}(x^\beta) + O(x \exp -c\sqrt{\log x})\right) \tag{8}$$

for all $x > \exp(C \log^2 q)$*. The constant implicit in each* $O(\cdot)$ *may depend on* C *but not on* q*.*

Proof: See the Exercises. $\Box$

Just how close can this β come to 1? We first show that very small $1 - \beta$ imply small $L(1, \chi)$. Since $L(1, \chi) = \int_\beta^1 L'(\sigma, \chi)\, d\sigma$, it is enough to prove an upper bound on $|L'(\sigma, \chi)|$ for σ near 1. We show:

Lemma. *There exists an absolute constant* C *such that* $|L'(\sigma, \chi)| < C \log^2 q$ *for any nontrivial Dirichlet character* $\chi \bmod q$ *and any* $\sigma \le 1$ *such that* $1 - \sigma \le 1/\log q$*.*

Proof: We may assume $q > 2$, so that the series $\sum_{n=1}^\infty \chi(n)(\log n) n^{-\sigma}$ for $-L'(\sigma, \chi)$ converges if $1 - \sigma \le 1/\log q$. Split this sum into $\sum_{n \le q} + \sum_{n > q}$. The first sum is $O(\log^2 q)$, because the n-th term has abolute value at most

$$\frac{n^{1-\sigma}}{n} \log n \le \frac{q^{1-\sigma}}{n} \log n \le \frac{e}{n} \log n.$$

The sum over $n > q$ can be bounded by partial summation together with the crude estimate $|\sum_q^N \chi(n)| < q$, yielding an upper bound $e \log q$, which is again $O(\log^2 q)$. $\Box$

Corollary. *If for some Dirichlet character* $\chi \bmod q$ *the* L*-series* $L(s, \chi)$ *has a zero* $\beta > 1 - (1/\log q)$ *then* $L(1, \chi) < C(1 - \beta) \log^2 q$*.*

But the Dirichlet class number formula for the quadratic number field corresponding to χ gives $L(1, \chi) \gg q^{-1/2}$. (We shall soon prove this directly.) Therefore

$$1 - \beta \gg \frac{1}{q^{1/2} \log^2 q}.$$

Siegel [1935] proved a much better inequality:

Theorem. *For each* $\epsilon > 0$ *there exists* $C_\epsilon > 0$ *such that*

$$L(1, \chi) > C_\epsilon q^{-\epsilon}$$

holds for all real Dirichlet characters $\chi \bmod q$*. Hence there exists* $C'_\epsilon > 0$ *such that any zero* β *of* $L(s, \chi)$ *satisfies*

$$1 - \beta > C'_\epsilon q^{-\epsilon}.$$

Proof: Let χ_1, χ_2 be different primitive real characters to moduli $q_1, q_2 > 1$, and let

$$\lambda = L(1,\chi_1)L(1,\chi_2)L(1,\chi_1\chi_2) = (s-1)F(s)\big|_{s=1},$$

with $F(s)$ as in (4). We shall prove that there exist universal constants $\theta < 1$ and $A, B, C > 0$ such that

$$F(s) > A - \frac{B\lambda}{1-s}(q_1 q_2)^{C(1-s)} \tag{9}$$

holds for all $s \in (\theta, 1)$. (Specifically, we can use $\theta = 9/10$ and $A = 1/2$, $C = 8$.) Assume (9) for the time being. Since $F(s)$ is positive for $s > 1$ and has a simple pole at $s = 1$, we have $F(\beta) \leq 0$ for any $\beta \in (\theta, 1)$ such that F has no zero in $(\theta, 1)$. Of course $F(\beta) \leq 0$ also holds if β is a zero of F. For such β we have

$$\lambda > \frac{A}{B}(1-\beta)(q_1 q_2)^{-C(1-\beta)}. \tag{10}$$

We shall fix χ_1 and β and use (10) to deduce a lower bound on $L(1,\chi_2)$ for all $\chi_2 \bmod q_2$ such that $q_2 \geq q_1$. If there is some real χ_1 such that $L(\beta_1, \chi_1) = 0$ for some $\beta_1 > 1 - (\epsilon/2C)$ then we use that character for χ_1 and the zero β_1 for β. Otherwise $F(s)$ never has a zero in $(1 - (\epsilon/2C), 1)$, so we choose χ_1 arbitrarily and β subject to $0 < 1 - \beta < \epsilon/2C$. Then for any primitive $\chi_2 \bmod q_2 \geq q_1$ we use (10), together with the upper bound $L(1,\chi) \ll \log q$ (see the Exercises), to find that

$$L(1,\chi_2) > c\, q_2^{-C(1-\beta)}/\log q_2,$$

with c depending only (but ineffectively!) on ϵ via χ_1 and β. Since $C(1-\beta) < \epsilon/2$, Siegel's theorem follows.

It remains to prove (9). Siegel originally showed this using class field theory; we follow the more direct approach of [Estermann 1948]. (See also [Chowla 1950] for another direct proof.)

Since $F(s)$ has a nonnegative Dirichlet series, its Taylor series about $s = 2$ is

$$F(s) = \sum_{m=0}^{\infty} b_m (2-s)^m$$

with $b_0 = F(2) > 1$ and all $b_m > 0$. Since F is entire except for a simple pole of residue λ at $s = 1$, we have the Taylor expansion

$$F(s) - \frac{\lambda}{s-1} = \sum_{m=0}^{\infty} (b_m - \lambda)(2-s)^m,$$

valid for all $s \in \mathbf{C}$. Consider this on $|s-2| = 3/2$. We have there the crude bounds $L(s,\chi_1) \ll q_1$, $L(s,\chi_2) \ll q_2$, $L(s,\chi_1\chi_2) \ll q_1 q_2$, and of course $\zeta(s)$ is bounded on $|s-2| = 3/2$. So, $F(s) \ll (q_1 q_2)^2$ on this circle, and thus the same is true of $F(s) - \lambda/(s-1)$. Hence

$$|b_m - \lambda| \ll (2/3)^m (q_1 q_2)^2.$$

For any fixed $\theta \in (1/2, 1)$ it follows that

$$\sum_{m=M}^{\infty} |b_m - \lambda| \, (2-s)^m \ll (q_1 q_2)^2 \Big(\frac{2}{3}(2-\theta)\Big)^M$$

holds for all $s \in (\theta, 1)$. Since $b_0 > 1$ and $b_m \geq 0$, we thus have

$$F(s) - \frac{\lambda}{s-1} \geq 1 - \lambda \frac{(2-s)^M - 1}{1-s} - O(q_1 q_2)^2 \Big(\frac{2}{3}(2-\theta)\Big)^M.$$

Let M be the largest integer such that the error estimate $O(q_1 q_2)^2((4-2\theta)/3)^M$ is $< 1/2$. Then

$$F(s) > \frac{1}{2} - \frac{\lambda}{1-s}(2-s)^M.$$

But

$$(2-s)^M = \exp(M \log(2-s)) < \exp M(1-s),$$

and $\exp M \ll (q_1 q_2)^{O(1)}$, which completes the proof of (9) and thus of Siegel's theorem.

Remarks

Unfortunately the constants C_ϵ and C'_ϵ in Siegel's theorem, unlike all such constants that we have encountered so far, are ineffective for every $\epsilon < 1/2$, and remain ineffective 80 years later. This is because we need more than one counterexample to reach a contradiction. What can be obtained effectively are constants C_ϵ such that $1 - \beta > C_\epsilon q^{-\epsilon}$ holds for every q and all real primitive characters mod q, with at most a single exception (q, χ, β). This β, if it exists, is called the "Siegel zero" or "Siegel-Landau zero". Note that a zero β of some $L(s, \chi)$ that violates the ERH may or may not qualify as a Siegel(-Landau) zero depending on the choice of ϵ and C_ϵ.

Dirichlet's class number formula relates $L(1, \chi)$ for real characters χ with the class number of quadratic number fields. Siegel's theorem and its refinements thus yield information on these class numbers. For instance, if χ is an odd character then the imaginary quadratic field $K = \mathbf{Q}(\sqrt{-q}\,)$ has a zeta function $\zeta_K(s)$ that factors into $\zeta(s) L(s, \chi)$. Let $h(K)$ be the class number of K. Siegel's theorem, together with Dirichlet's formula, yields the estimate $h(K) \gg_\epsilon q^{1/2-\epsilon}$. In particular, $h(K) > 1$ for all but finitely many K. But this does not reduce the determination of all such K to a finite computation because the implied constant cannot be made effective. Gauss had already conjectured in 1801 (in terms of binary quadratic forms, see *Disq. Arith.*, §303) that $h(K) = 1$ only for $K = \mathbf{Q}(\sqrt{-q}\,)$ with

$$q = 3,\ 4,\ 7,\ 8,\ 11,\ 19,\ 43,\ 67,\ 163.$$

But the closest that Siegel's method can bring us to this conjecture is the theorem that there is at most one further such q. Heilbronn and Linfoot showed this a year before Siegel's theorem [HL 1934] using a closely related method [Heilbronn 1934].

It was only with much further effort that Heegner ([1952], corrected by Deuring [1968]), and later Baker [1966] and Stark [1967] (working independently and using different approaches), proved Gauss's conjecture, at last exorcising the "tenth discriminant". None of these approaches yields an effective lower bound on $h(K)$ that grows without limit as $q \to \infty$. Such a bound was finally obtained by Goldfeld, Gross, and Zagier ([Goldfeld 1976], [GZ 1986]), but it grows much more slowly than $q^{1/2-\epsilon}$; namely, $h(K) > c \log q / \log \log q$ (and $h(K) > c' \log q$ for prime q). Even this was a major breakthrough that combined difficult algebraic and analytic techniques. See [Goldfeld 1986] for an overview.

Exercises

1. Complete the derivation of (5,6,7,8) from our (nearly) zero-free region for Dirichlet L-functions.

2. Show that for each $\epsilon > 0$ there exists C such that whenever $\gcd(a, q) = 1$ we have

$$|\varphi(q)\, \psi(x, a \bmod q) - x| < \epsilon x$$

for all $x > q^C$, unless there is a Dirichlet character $\chi \bmod q$ for which $L(s, \chi)$ has an exceptional zero β, in which case

$$|\varphi(q)\, \psi(x, a \bmod q) - x - \chi(a)(x^\beta/\beta)| < \epsilon x$$

for all $x > q^C$. [NB $q^C = \exp(C \log q)$.] Use this argument, together with the fact that $\psi(x, a \bmod q) \geq 0$, to give an alternative proof of Landau's theorem (at most one exceptional zero for any character mod q). Also, obtain the corresponding estimates for $\pi(x, a \bmod q)$, and deduce that there is an absolute constant C such that if there is no exceptional zero then there exists a prime $p \equiv a \bmod q$ with $p < q^C$. (Linnik [1944] showed this unconditionally, by showing that very small values of $1 - \beta$ force the other zeros farther away from the line $\sigma = 1$.)

3. Check that $|L'(\sigma, \chi)| \ll_A \log^2 q$ holds in any interval $0 \leq 1 - \sigma \leq A/\log q$. How does the implied constant depend on A? Use the same method to show that also $|L(\sigma, \chi)| \ll_A \log q$ in the same interval, and in particular at $\sigma = 1$. What bound can you obtain on the higher derivatives of $L(\sigma, \chi)$ for σ near 1?

4. Prove that for every $\epsilon > 0$ there exist effective positive constants A, c such that if $L(1, \chi) < A q^{-\epsilon}$ for some primitive real Dirichlet character $\chi \bmod q$ then $L(\beta, \chi) = 0$ for some β such that $1 - \beta < c/\log q$. [Use the fact that

$$L(1, \chi) = L(\sigma, \chi) \exp \int_1^\sigma -\frac{L'}{L}(s, \chi)\, ds,$$

or work directly with the Hadamard product formula for $L(s, \chi)$. This Exercise, which shows that small $L(1, \chi)$ implies small $1 - \beta$, may be regarded as a qualitative converse of our computation showing that small $1-\beta$ implies small $L(1,\chi)$.]

References

[Baker 1966] Baker, A.: Linear forms in the logarithms of algebraic numbers, *Mathematika* **13** (1966), 204–216.

[Chowla 1950] Chowla, S.: A new proof of a theorem of Siegel. *Annals of Math.* (2) **51** (1950), 120–122.

[Deuring 1968] Deuring, M.: Imaginäre quadratische Zahlkörper mit der Klassenzahl Eins, *Invent. Math.* **5** (1968), 169–179.

[Estermann 1948] Estermann, T.: On Dirichlet's L functions, *J. London Math. Soc.* **23** (1948), 275–279.

[Goldfeld 1976] Goldfeld, D.M.: The class number of quadratic fields and the conjectures of Birch and Swinnerton-Dyer, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **3** (1976) #4, 624–663.

[Goldfeld 1986] Goldfeld, D.M.: Gauss's class number problem for imaginary quadratic fields, *Bull. Amer. Math. Soc.* (N.S.) **13** (1985) #1, 23–37.

[GZ 1986] Gross, B.H., Zagier, D.B.: Heegner points and derivatives of L-series, *Invent. Math.* **84** (1986) #2, 225–320.

[Heegner 1952] Heegner, K.: Diophantische Analysis und Modulfunktionen, *Math. Z.* **56** (1952), 227–253.

[Heilbronn 1934] Heilbronn, H.: On the Class-number in Imaginary Quadratic Fields, *Quarterly J. Math.* **5** (1934), 150–160.

[HL 1934] Heilbronn, H., Linfoot, E.H.: On the imaginary quadratic corpora of class-number one, *Quarterly J. Math.* **5** (1934), 293–301;

[Landau 1918] Landau, E: Über die Classenzahl imaginär-quadratischer Zahlkörper, *Göttinger Nachrichten* 1918, 285–295. [= pages 150–160 of *Collected Works* Vol.VII, Essen: Thales, 1985.] [QA3.L24]

[Linnik 1944] Linnik, U.V.: On the Least Prime in an Arithmetic Progression. I. The Basic Theorem; II. The Deuring-Heilbronn Phenomenon. *Mat. Sbornik* (N.S.) **15 (57)** (1944), 139–178 and 347–368.

[Siegel 1935] Siegel, C.L.: Über die Classenzahl quadratischer Zahlkörper, *Acta Arith.* **1** (1935), 83–86 [= pages 406–409 (#21) of *Gesammelte Abhandlungen* (Collected Works) Vol.I, Berlin: Springer, 1966.] [O 9.66.2 / QA3.S56]

[Stark 1967] Stark, H.M.: A complete determination of the complex quadratic fields of class-number one, *Michigan Math. J.* **14** (1967), 1–27.