

## Math 229: Introduction to Analytic Number Theory

The contour integral formula for  $\psi(x)$

We now have several examples of Dirichlet series, that is, series of the form<sup>1</sup>

$$F(s) = \sum_{n=1}^{\infty} a_n n^{-s} \tag{1}$$

from which we want to extract information about the growth of  $\sum_{n < x} a_n$  as  $x \rightarrow \infty$ . The key to this is a contour integral. We regard  $F(s)$  as a function of a complex variable  $s = \sigma + it$ . For real  $y > 0$  we have seen already that  $|y^{-s}| = y^{-\sigma}$ . Thus if the sum (1) converges absolutely<sup>2</sup> for some real  $\sigma_0$ , then it converges uniformly and absolutely to an analytic function on the half-plane  $\text{Re}(s) \geq \sigma_0$ ; and if the sum converges absolutely for all real  $s > \sigma_0$ , then it converges absolutely to an analytic function on the half-plane  $\text{Re}(s) > \sigma_0$ . Now for  $y > 0$  and  $c > 0$  we have

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^s \frac{ds}{s} = \begin{cases} 1, & \text{if } y > 1; \\ \frac{1}{2}, & \text{if } y = 1; \\ 0, & \text{if } y < 1, \end{cases} \tag{2}$$

in the following sense: the contour of integration is the vertical line  $\text{Re}(s) = c$ , and since the integral is then not absolutely convergent it is regarded as a principal value:

$$\int_{c-i\infty}^{c+i\infty} f(s) ds := \lim_{T \rightarrow \infty} \int_{c-iT}^{c+iT} f(s) ds.$$

Thus interpreted, (2) is an easy exercise in contour integration for  $y \neq 1$ , and an elementary manipulation of  $\log s$  for  $y = 1$ . So we expect that if (1) converges absolutely in  $\text{Re}(s) > \sigma_0$  then

$$\sum_{n < x} a_n = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^s F(s) \frac{ds}{s} \tag{3}$$

for any  $c > \sigma_0$ , using the principal value of the integral and adding  $a_x/2$  to the sum if  $x$  happens to be an integer. But getting from (1) and (2) to (3) involves interchanging an infinite sum with a conditionally convergent integral, which is not in general legitimate. Thus we replace  $\int_{c-i\infty}^{c+i\infty}$  by  $\int_{c-iT}^{c+iT}$ , which legitimizes the manipulation but introduces an error term into (2). We estimate this error term as follows:

**Lemma.** For  $y, c, T > 0$  we have

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} y^s \frac{ds}{s} = \begin{cases} 1 + O(y^c \min(1, \frac{1}{T|\log y|})), & \text{if } y \geq 1; \\ O(y^c \min(1, \frac{1}{T|\log y|})), & \text{if } y \leq 1, \end{cases} \tag{4}$$

<sup>1</sup>As noted in [Serre 1973, Ch.6 §2], everything works just as well with “Dirichlet series”  $\sum_{k=0}^{\infty} a_k n_k^{-s}$ , where  $n_k$  are positive reals such that  $n_k \rightarrow \infty$  as  $k \rightarrow \infty$ . In that more general setting we would seek to estimate  $\sum_{n_k < x} a_k$  as  $x \rightarrow \infty$ .

<sup>2</sup>We shall see later that the same results hold if absolute convergence is replaced by conditional convergence throughout. For example, for every nonprincipal character  $\chi$  the series for  $L(s, \chi)$  converges uniformly in the half-plane  $\text{Re}(s) > \sigma_0$  for each positive  $\sigma_0$ .

the implied  $O$ -constant being effective and uniform in  $y, c, T$ .

(In fact the error's magnitude is less than both  $y^c$  and  $y^c/(\pi T |\log y|)$ . Of course if  $y$  equals 1 then the error term is regarded as  $O(1)$  and is valid for both approximations 0, 1 to the integral.)

*Proof:* Complete the contour of integration to a rectangle extending to real part  $-M$  if  $y \geq 1$  or  $+M$  if  $y \leq 1$ . The resulting contour integral is 1 or 0 respectively by the residue theorem. We may let  $M \rightarrow \infty$  and bound the magnitude of each horizontal integral by  $(2\pi T)^{-1} \int_0^\infty y^{c \pm r} dr$ ; this gives the estimate  $y^c/(\pi T |\log y|)$ . Using a circular arc centered at the origin instead of a rectangle yields the same residue with a remainder of absolute value  $< y^c$ .  $\square$

This Lemma will let us approximate  $\sum_{n < x} a_n$  by  $(2\pi i)^{-1} \int_{c-iT}^{c+iT} x^s F(s) ds/s$ . We shall eventually choose some  $T$  and exploit the analytic continuation of  $F$  to shift the contour of integration past the region of absolute convergence to obtain nontrivial estimates.

Which  $F$  should we choose? Consider for instance  $\zeta(s)$ . We have in effect seen already that if we take  $F(s) = \log \zeta(s)$  then the sum of the resulting  $a_n$  over  $n < x$  closely approximates  $\pi(x)$ . Unfortunately, while  $\zeta(s)$  continues meromorphically to  $\sigma \leq 1$ , its logarithm does not: it has essential logarithmic singularities at the pole  $s = 1$  and at zeros of  $\zeta(s)$  to be described later. So we use the *logarithmic derivative* of  $\zeta(s)$  instead, which at each pole or zero of  $\zeta$  has a simple pole with a known residue and thus a predictable effect on our contour integral.

What are the coefficients  $a_n$  for this logarithmic derivative? It is convenient to use not  $\zeta'/\zeta$  but  $-\zeta'/\zeta$ , which has positive coefficients. Using the Euler product we find

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{d}{ds} \log(1 - p^{-s}) = \sum_p \log p \frac{p^{-s}}{1 - p^{-s}} = \sum_p \log p \sum_{k=1}^{\infty} p^{-ks}.$$

That is,

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \Lambda(n) n^{-s}.$$

So the coefficient of  $n^{-s}$  is none other than the von Mangoldt function which arose in the factorization of  $x!$ . Hence our contour integral

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} -\frac{\zeta'}{\zeta}(s) x^s \frac{ds}{s} \quad (c > 1)$$

approximates  $\psi(x)$ . The error can be estimated by our Lemma (4): since  $|\Lambda(n)| \leq \log n$ , the error is of order at most

$$\sum_{n=1}^{\infty} (x/n)^c \log n \cdot \min\left(1, \frac{1}{T |\log(x/n)|}\right)$$

which is  $O(T^{-1} x^c (\log^2 x + \frac{1}{c-1}))$  provided  $1 < T < x$ . (See the Exercises below.)

Now the factor  $x^c$  appearing in the error estimate exceeds  $x$  for any  $c > 1$ . To ensure that  $x^c \ll x$ , we fix some  $A > 0$  and take  $c = 1 + (A/\log x)$ ; this also makes the term  $1/(c-1)$  negligible compared with  $\log^2 x$ . We then find:

$$\psi(x) = \frac{1}{2\pi i} \int_{1+\frac{A}{\log x}-iT}^{1+\frac{A}{\log x}+iT} -\frac{\zeta'}{\zeta}(s) x^s \frac{ds}{s} + O_A\left(\frac{x \log^2 x}{T}\right). \quad (5)$$

Similarly for any Dirichlet character  $\chi$  we obtain a formula for

$$\psi(x, \chi) := \sum_{n < x} \chi(n) \Lambda(n)$$

by replacing  $\zeta(s)$  in (5) by  $L(s, \chi)$ .

To make use of this, we want to shift the line of integration to the left, where  $|x^s|$  is smaller. As we do this, we shall encounter a pole of  $\zeta(s)$  at  $s = 1$  and zeros of  $\zeta(s)$  (or  $L(s, \chi)$ ) at other values of  $s$ , leading us to estimate  $|\zeta'(s)/\zeta(s)|$  (or  $|L'(s, \chi)/L(s, \chi)|$ ) for  $s$  on the resulting contour. **This is why we are interested in the analytic continuation of  $\zeta(s)$  and of  $L(s, \chi)$ , and in their zeros.** We investigate these matters next.

### Remarks

We can already surmise that  $\psi(x)$  will be approximated by  $x - \sum_{\rho} x^{\rho}/\rho$ , the sum running over zeros  $\rho$  of  $\zeta(s)$  counted with multiplicity, and thus that the Prime Number Theorem is tantamount to the nonvanishing of  $\zeta(s)$  on  $\text{Re}(s) = 1$ . The fact that  $\zeta(1+it) \neq 0$  is also the key step in various “elementary” proofs of the Prime Number Theorem such as [Newman 1980] (see also [Zagier 1997]). Likewise for  $L(1+it, \chi)$  and the asymptotic formula for  $\pi(x, a \bmod q)$ .

The formula for  $\psi(x)$  as a contour integral can be viewed as an instance of the inverse Mellin transform. Suppose  $F(s)$  is a generalized Dirichlet series  $\sum_{k=0}^{\infty} a_k n_k^{-s}$ , converging for  $\text{Re}(s) > \sigma_0$ . Let  $A(x) = \sum_{n_k < x} a_k$ , and assume that  $A(x) \rightarrow \infty$  as  $x \rightarrow \infty$ . In particular,  $\sigma_0 \geq 0$ . Now

$$F(s) = \int_0^{\infty} x^{-s} dA(x) = s \int_0^{\infty} x^{-s} A(x) \frac{dx}{x},$$

so  $F(s)/s$  is the Mellin transform of  $A(x)$  evaluated at  $-s$ . Thus we expect that

$$A(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^s F(s) \frac{ds}{s}$$

for  $c > \sigma_0$ . Due to the discontinuities of  $A(x)$  at  $x = n_k$ , this integral cannot converge absolutely, but its principal value does equal  $A(x)$  at all  $x \notin \{n_k\}$ .

### Exercises

1. Verify that the error

$$\sum_{n=1}^{\infty} (x/n)^c \log n \cdot \min\left(1, \frac{1}{T|\log(x/n)|}\right)$$

in our approximation of  $\psi(x)$  is  $O(T^{-1}x^c(\log^2 x + \frac{1}{c-1}))$  provided  $1 < T < x$ . Explain why the bound need not hold if  $T$  is large compared to  $x$ .

2. Use (4) to show that nevertheless  $\psi(x)$  is given by the principal value integral

$$\psi(x) = \lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} -\frac{\zeta'}{\zeta}(s) x^s \frac{ds}{s} \quad (6)$$

for all  $x, c > 1$  such that  $x \notin \mathbf{Z}$ . What is the right-hand side of (6) when  $c > 1$  and  $x$  is a positive integer?

3. Show that  $\sum_{n=1}^{\infty} \mu(n)n^{-s} = 1/\zeta(s)$ , with  $\mu$  being the Möbius function defined in the previous set of exercises. Deduce an integral formula for  $\sum_{n < x} \mu(n)$  analogous to (6), and an approximate integral formula analogous to (5) but with error only  $O(T^{-1}x \log x)$  instead of  $O(T^{-1}x \log^2 x)$ .

### References

[Newman 1980] Newman, D.J.: Simple Analytic Proof of the Prime Number Theorem, *Amer. Math. Monthly* **87** (1980), 693–696.

[Serre 1973] Serre, J.-P.: *A Course in Arithmetic*. New York: Springer, 1973 (Graduate Texts in Mathematics **7**).

[Zagier 1997] Zagier, D.: Newman's Short Proof of the Prime Number Theorem, *Amer. Math. Monthly* **104** (1994), 705–708.