Math 229: Introduction to Analytic Number Theory

An upper bound on the coefficients of a $\mathrm{PSL}_2(\mathbf{Z})$ cusp form

Fix an integer $k > 1$, and let M_k^0 be the space of cusp forms of weight $2k$ for $G = \mathrm{PSL}_2(\mathbf{Z})$. It is known that this is a finite-dimensional vector space. Moreover it carries a Hermitian (*Petersson*) pairing [Serre 1973, VII 5.6.1 (p.105)]:

$$\langle f, g \rangle = \iint_{\mathcal{H}/G} f(z)\overline{g(z)} y^{2k-2}\, dx\, dy.$$

Now for each integer $n > 0$ the map taking a cusp form $f = \sum_{m=1}^{\infty} a_m q^m$ to a_n is a linear functional on M_k^0. Thus there is a unique $P_n \in M_k^0$ that represents this functional:

$$\langle f, P_n \rangle = a_n(f)$$

for all $f \in M_k^0$. Moreover we have:

Lemma. *The P_n for $n \leq \dim M_k^0$ constitute a basis for M_k^0.*

Proof: The orthogonal complement of the linear span of these P_n is the subspace of $f \in M_k^0$ whose first $\dim M_k^0$ coefficients vanish, and it is known that 0 is the only such f. $\square$

Therefore an upper bound $a_r(P_n) \ll_n r^\theta$ for all $n \leq \dim M_k^0$ will yield the bound $a_r(f) \ll_f r^\theta$ for all $f \in M_k^0$.

Remarkably we can obtain P_n and its q-expansion in an explicit enough form (a "Poincaré series") to obtain such an inequality for all $\theta > k - \frac{1}{4}$ — and the proof uses Weil's bound [Weil 1948] on Kloosterman sums! (See for instance [Selberg 1965, §3]; thanks to Peter Sarnak for this reference and for first introducing me to this approach. It is now known that in fact the correct θ is $k - \frac{1}{2} + \epsilon$, but Deligne's proof of this is quite deep, and is not as generally applicable: the Poincaré-series method still yields the sharpest bounds known for some other kinds of modular forms.)

We begin by observing that for any $f(z) = \sum_{n=1}^{\infty} a_n q^n$ the coefficient a_n may be isolated from the absolutely convergent double integral

$$\iint_{0<x<1} \overline{e^{2\pi i n z}} f(z) y^{2k-2}\, dx\, dy = a_n \int_0^\infty e^{-4\pi n y} y^{2k-2}\, dy = \frac{(2k-2)!}{(4\pi n)^{2k-1}} a_n.$$

Now the region we're integrating over is a fundamental domain for the action of $\langle T \rangle$ on $\mathcal{H}$. We decompose this as the union (with only boundary overlaps) of G-images of the fundamental domain for the action of G. That is, we split up the integral as

$$\sum_g \iint_D \overline{e^{2\pi i n g(z)}} f(g(z)) y^{2k}(g(z)) \frac{dx\, dy}{y^2}$$

where D is a fundamental domain for $\mathcal{H}/G$ and g ranges over coset representatives of $\langle T \rangle \backslash G$. But these cosets amount to coprime pairs (c, d) of integers with

$c > 0$ or $c = 0, d = 1$. Moreover, we have for $g(z) = (az + b)/(cz + d)$

$$f(g(z))y^{2k}(g(z)) = (cz + d)^{2k}y^{2k}(g(z))f(z) = f(z)\, y^{2k}(z)/(c\bar{z} + d)^{2k}.$$

So, we find

$$\frac{(2k-2)!}{(4\pi n)^{2k-1}}a_n = \iint_D f(z)\, \overline{\sum_{c,d}\sum (cz+d)^{-2k}\exp\left(2\pi in\frac{az+b}{cz+d}\right)}\, y^{2k-2}\, dx\, dy.$$

Therefore the double sum is $(4\pi n)^{1-2k}(2k-2)!P_n$, provided we can show that it is in fact a cusp form — which, however, is surprisingly easy. [To do away with the requirement that $c > 0$ or $(c, d) = (0, 1)$ we may sum over all coprime pairs (c, d), then divide by 2. The sum converges absolutely because it is dominated by the sum defining the Eisenstein series E_k: the factors $e(ng(z))$ all have absolute value < 1.] We thus have:

$$\frac{(4\pi n)^{2k-1}}{(2k-2)!}P_n(z) = \sum_{c,d}\sum (cz+d)^{-2k}\exp\left(2\pi in\frac{az+b}{cz+d}\right). \tag{1}$$

(Note that the exponential factor does not depend on the choice of $a, b \in \mathbf{Z}$ such that $ad - bc = 1$.)

We next determine the q-expansion of the Poincaré series P_n. The term $(c, d) = (0, 1)$ contributes q^n to the sum. We group the remaining terms according to c and d mod c. [The existence of a q-expansion is equivalent to T-invariance, so to obtain the q-expansion we collect the $(az+b)/(cz+d)$ into $\langle T\rangle$-orbits, which is to say that we now consider P_n as a sum over the double coset space $\langle T\rangle\backslash G/\langle T\rangle$.] Fix coprime c, d_0 with $c > 0$, and a_0, b_0 such that $a_0d_0 - b_0c = 1$. Then the terms of the sum (1) with $d \equiv d_0 \bmod c$ have $(a, b, c, d) = (a_0, b_0 + ma_0, c, d_0 + mc)$ for $m \in \mathbf{Z}$, and thus contribute

$$\sum_{m\in\mathbf{Z}} (c(z+m)+d_0)^{-2k}\exp\left(2\pi in\frac{a_0(z+m)+b_0}{c(z+m)+d_0}\right).$$

By Poisson summation this is $\sum_{r\in\mathbf{Z}} u_r$ where

$$u_r := \int_{-\infty}^{\infty} (c(z+t)+d_0)^{-2k}\exp\left(2\pi in\frac{a_0(z+t)+b_0}{c(z+t)+d_0}\right)e^{-2\pi irt}\, dt. \tag{2}$$

If $r \le 0$ the integrand extends to a holomorphic function on $\operatorname{Im} t \ge 0$ bounded by $|c(z + t) + d_0|^{-2k} \ll |t|^{-2k}$, and thus the integral vanishes by a standard contour integration. So we need only consider (2) for $r > 0$. In that case, let $w = z + t + (d_0/c)$. We then find

$$u_r = c^{-2k}q^r e^{2\pi i(na_0+rd_0)/c}\int_C e^{-2\pi i(rw+n/c^2w)}w^{-2k}\, dw, \tag{3}$$

with the contour of integration C passing above the essential singularity at $w = 0$. Note that the integral depends only on n, r, c but not on d_0; the dependence on d_0 is entirely contained in the factor $e^{2\pi i(a_0+rd_0)/c}$, in which a_0 is the multiplicative inverse of d_0 mod c. Summing over c, d_0 we thus find that for $r \neq n$ the q^r coefficient of P_n is

$$\frac{(2k-2)!}{(4\pi n)^{2k-1}} \sum_{c=1}^{\infty} c^{-2k} K_c(n,r) \int_C e^{-2\pi i (rw + n/c^2 w)} w^{-2k}\, dw.$$

The Kloosterman sum $K_c(n,r)$ is $O_n(c^{1/2+\epsilon})$ as seen already. The integral is essentially a Bessel function: let $-2\pi i r w = v$ to get

$$(-2\pi i r)^{2k-1} \int_{-iC} \exp\Bigl(v - \frac{4\pi^2 rn}{c^2 v}\Bigr) v^{-2k}\, dv = (-1)^k 2\pi (c\sqrt{r/n})^{2k-1} J_{2k-1}(4\pi\sqrt{rn}/c)$$

([GR 1980, 8.412 2.], taken from [Watson 1944]). So the q^r coefficient of P_n is

$$\ll_{k,n,\epsilon} \sum_{c=1}^{\infty} c^{\epsilon - \frac{1}{2}} r^{k-\frac{1}{2}} |J_{2k-1}(4\pi\sqrt{rn}/c)|. \tag{4}$$

Now it is known ([GR 1980, 8.451], again from [Watson 1944]) that $J_{2k-1}(x) \ll x^{-1/2}$ for large $x > 0$ while $J_{2k-1}(x) \sim C_k x^{2k-1}$ for small $x > 0$. Splitting the sum in (4) around $c = \sqrt{r}$ we find that both parts are $\ll r^{1/4+\epsilon}$. Thus each P_n has q^r coefficient $O_\epsilon(r^{k-1/4+\epsilon})$ as claimed, and we are done. $\Box$ (whew!)

Exercises

1. Let $f = \sum_{n>0} a_n q^n$ be a cusp form of weight $2k$. Use the boundedness of $y^{2k}|f(z)|^2$ to prove that $\sum_{n<N} |a_n|^2 \ll_f N^{2k}$. [In other words $a_n \ll n^{k-1/2}$ in mean square. Note that Hecke's estimate $|a_n| \ll_f n^k$ follows immediately.]

2. Let $f = \sum_{n>0} a_n q^n$ be a cusp form of weight $2k$, and let $L_f(s) = \sum_n a_n n^{-s}$ be the associated L-function (called $\Phi_f(s)$ in [Serre 1973, p.103]). Use the integral representation of L_f to prove that $L_f(\sigma + it) \ll_{f,\sigma} |t|^{\theta_k(\sigma)}$ for some $\theta_k(\sigma) < \infty$. How small a $\theta_k(\sigma)$ can you obtain? [As usual, it is conjectured *á la* Lindelöf that $L_f(\sigma+it) \ll_{f,\epsilon} |t|^\epsilon$ for all $\sigma \geq k$.]

3. Verify that in fact $P_n \in M_k^0$.

4. Verify that our final estimate on (4) follows from the J_{2k-1} asymptotics cited. Since $|K_c(n,r)|/\sqrt{nc}$ is actually $\leq \prod_{p|c} 2$, which in turn is bounded by the number of factors of c, we can make the r^ϵ factor more precise; show that in fact $\log r$ suffices, i.e., the q^r coefficient of a cusp form of weight $2k$ is $O(r^{k-1/4}\log r)$.

5. For each even $k = 2, 4, 6, \ldots$ there is a unique $f = \sum_{n=0}^{\infty} a_n q^n \in M_k$ of the form $1 + O(q^{\lfloor k/6 \rfloor + 1})$, i.e., such that $a_0 = 1$ and $a_1 = a_2 = \cdots = a_{\lfloor k/6 \rfloor} = 0$. (Why?) Prove that $a_{\lfloor k/6 \rfloor + 1} > 0$. [This is a bit tricky, requiring the residue formula and the fact that $1/\Delta = q^{-1} + 24q + 324q + 3200q^2 + \cdots$ has positive coefficients — a fact that can be deduced from the Jacobi product for Δ.] Conclude that an even unimodular lattice in dimension $4k$ has a vector of norm at most $2(\lfloor k/6 \rfloor + 1)$.

Can the minimal norm be that large? Such lattices exist for several small k, including $k = 2, 4, 6, \ldots, 16$, but it is known that for all but finitely many k the minimal norm is always strictly smaller, indeed $< 2(\lfloor k/6 \rfloor - \delta)$ once $k > k(\delta)$ for some effectively computable $k(\delta)$. This is shown by proving that there is no suitable modular form all of whose coefficients are nonnegative. Still many open questions remain; for instance it is not even known whether there is an even unimodular lattice of dimension 72 and minimal norm 8. How many minimal vectors would such a lattice have? See [CS 1993] for more along these lines, especially p.194 and thereabouts.

References

[CS 1993] Conway, J.H., Sloane, N.J.A.: *Sphere Packings, Lattices and Groups.* New York: Springer 1993.

[GR 1980] Gradshteyn, I.S., Ryzhik, I.M.: *Table of Integrals, Series, and Products.* New York: Academic Press 1980. [D 9.80.1 / BASEMENT REFERENCE QA55.G6613]

[Selberg 1965] Selberg, A.: On the estimation of Fourier coefficients of modular forms, #33 (506–520) in his *Collected Papers I.* [O 9.89.2 (I)]

[Serre 1973] Serre, J.-P.: *A Course in Arithmetic.* New York: Springer, 1973 (GTM **7**). [AB 9.70.4 (reserve case) / QA243.S4713]

[Watson 1944] Watson, G.N.: *A treatise on the theory of Bessel functions* (2nd ed.) Bew York: Macmillan 1944. [HA 9.44.6 / QA408.W2]

[Weil 1948] Weil, A.: On some exponential sums. Item 1948c (pages 386–389) in his *Collected Papers I.* [O 9.79.1 (I) / QA3.W43]