

Math 21b: Second Midterm

Please show your work for all problems except for the True/False questions on the last page. If you need additional space, feel free to use the backs of the pages (please make a note if you do so, so we know to look). Each of the five questions is worth ten points. No calculators, notes, books, or any other aids are allowed.

Please don't write on this front page (except for your name) as we will use it to record grades.

Name:

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Total	

1. Let V be the subspace of \mathbb{R}^4 with basis

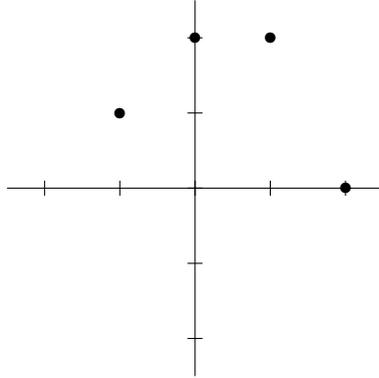
$$\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 3 \\ 9 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 6 \\ 3 \end{bmatrix}.$$

(a) [4 pts.] Apply Gram-Schmidt to this basis to find an orthonormal basis (\vec{w}_1, \vec{w}_2) for V . Check that in fact $\vec{w}_1 \cdot \vec{w}_2 = 0$.

(b) [2 pts.] What are the coordinates of \vec{v}_1 and \vec{v}_2 with respect to the basis (\vec{w}_1, \vec{w}_2) of V ?

(c) [4 pts.] Find the orthogonal projection \vec{e}_1^{\parallel} of the standard vector \vec{e}_1 onto V . Check your work by verifying that \vec{e}_1^{\perp} is orthogonal to \vec{v}_1 and \vec{v}_2 .

2. [10 pts.] Which of the following is the equation for the best fit of a line through the four points $(-1, 1)$, $(0, 2)$, $(1, 2)$, $(2, 0)$, as determined by the least squares method? Show your work!



- (a) $y = \frac{4}{3} + \frac{1}{3}x$
- (b) $y = \frac{13}{10} - \frac{1}{10}x$
- (c) $y = \frac{14}{10} - \frac{3}{10}x$
- (d) $y = \frac{5}{4}$
- (e) $y = \frac{10}{6} - \frac{1}{6}x$

3. For a scalar c , let A be the matrix

$$\begin{bmatrix} 1 & 0 & 0 & c^4 \\ 1 & c^2 - 2 & 0 & 0 \\ 0 & 1 & c^2 - 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

(a) [4 pts.] Find the determinant of A .

(b) [4 pts.] For which c is the matrix A invertible?

(c) [2 pts.] For which c is $\det(AA^T) = 0$?

4. The sequence a_0, a_1, a_2, \dots is given by a recursion

$$a_{n+1} = \frac{1}{2}(a_n + a_{n-1}).$$

The dynamical system describing this sequence is given by an equation $\vec{x}(t+1) = A\vec{x}(t)$, where

$$\vec{x}(t) = \begin{bmatrix} a_{t+1} \\ a_t \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}.$$

We shall take $a_0 = 3$ and $a_1 = 24$, so that $\vec{x}(0) = \begin{bmatrix} 24 \\ 3 \end{bmatrix}$.

(a) [2 pts.] Compute the eigenvalues of A .

(b) [2 pts.] Compute an eigenvector for each of the eigenvalues you obtained in (a).

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(c) [2 pts.] Give a closed form formula for $\vec{x}(t)$.

(d) [2 pts.] What is $\lim_{t \rightarrow \infty} \vec{x}(t)$?

(e) [2 pts.] Briefly describe in words the long-term behavior of the sequence a_0, a_1, a_2, \dots , and sketch the phase portrait of the dynamical system.

5. For each of the following 10 assertions, circle **T** if the assertion is true, and circle **F** if the assertion is false. Each is worth one point. For this question only, there is no need to justify your answers.

T F a) If matrices A and B are similar, and A is invertible, then B must be invertible as well.

T F b) If \vec{x} and \vec{y} are vectors in \mathbb{R}^{10} such that $\|\vec{x}\| = 3$, $\|\vec{y}\| = 4$, and $\|\vec{x} + \vec{y}\| = 5$, then \vec{x} and \vec{y} are orthogonal.

T F c) If A is a symmetric matrix, \vec{v} is a vector in the image of A , and \vec{w} is a vector in the kernel of A , then the equation $\vec{v} \cdot \vec{w} = 0$ must hold.

T F d) There exists a 3×3 orthogonal matrix one of whose entries is 3.

T F e) There exists an invertible 3×3 matrix A that is similar to $3A$.

T F f) An orthogonal matrix can have eigenvalue $1/2$.

T F g) If a matrix A has two eigenvalues λ_1 and λ_2 whose eigenspaces $E_{\lambda_1}, E_{\lambda_2}$ satisfy $E_{\lambda_1} = E_{\lambda_2}^\perp$ then A is diagonalizable.

T F h) There exists a matrix A satisfying $A^2 - 4A + 3I_n = 0$ and having the eigenvalue 2.

T F i) If 1 and 2 are eigenvalues of a 3×3 matrix A then A is diagonalizable.

T F j) There exists a 6×6 matrix A such that $\text{tr}(A) = 0$ and A has 6 positive eigenvalues.