

Solution Midterm 1, problem 1

Math 21b

You are given the linear system

$$\begin{aligned}x - 2kz &= 0 \\x + 2y + 6z &= 2 \\2z - kx &= 1\end{aligned}$$

(a) For $k = 2$, the augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 1 & 2 & 6 & 2 \\ -2 & 0 & 2 & 1 \end{array} \right)$$

Subtract row 1 from row 2 and add 2 times row 1 to row 3.

$$\left(\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 2 & 10 & 2 \\ 0 & 0 & -6 & 1 \end{array} \right)$$

Divide row 2 by 2 and row 3 by -6.

$$\left(\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & 5 & 1 \\ 0 & 0 & 1 & -1/6 \end{array} \right)$$

Add 4 times row 3 to row 1 and subtract 5 times row 3 to row 2.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -4/6 \\ 0 & 1 & 0 & 11/6 \\ 0 & 0 & 1 & -1/6 \end{array} \right)$$

This gives solution $x = -2/3$, $y = 11/6$ and $z = -1/6$.

To check this is correct simply substitute into the equations: $-2/3 + 4/6 = 0$ and $-2/3 + 22/6 - 6/6 = 2$ and $-2/6 - (-1)/6 = 1$.

(b) The linear system of equations will not have a unique solution if it has infinitely many solutions or no solutions. This occurs when the rank of the coefficient matrix is strictly less than 3. The coefficient matrix is:

$$\begin{pmatrix} 1 & 0 & -2k \\ 1 & 2 & 6 \\ -k & 0 & 2 \end{pmatrix}$$

Subtract row 1 from row 2 and add k times row 1 to row 3.

$$\begin{pmatrix} 1 & 0 & -2k \\ 0 & 2 & 6 + 2k \\ 0 & 0 & 2 - 2k^2 \end{pmatrix}$$

Divide row 2 by 2.

$$\begin{pmatrix} 1 & 0 & -2k \\ 0 & 1 & 3 + k \\ 0 & 0 & 2 - 2k^2 \end{pmatrix}$$

The matrix has rank less than 3 only if $2 - 2k^2 = 0$. That is $k = \pm 1$.

(c) For $k = \pm 1$ the augmented matrix has last row $(0 \ 0 \ 0 \ | \ 1)$. Hence for these values of k , the system is inconsistent and has no solution. There are no values of k which yield a system with infinitely many solutions.