

Math 21b

FINAL EXAMINATION

Please show your work for all problems except for the True/False questions on the last page. If you need additional space, feel free to use the backs of the pages (please make a note if you do so, so we know to look). Each of the nine questions is worth ten points. No calculators, notes, books, or any other aids are allowed.

Please don't write on this front page (except for your name and class time) as we will use it to record grades.

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Section:

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1. A matrix M is of the form

$$\begin{bmatrix} 0 & * & 5 & * \\ * & * & * & * \end{bmatrix}.$$

where, as usual, the $*$'s denote unknown and possibly different real numbers.

(a) [4 points] Given that M is in **row-reduced echelon form**, find all possible M , and explain why there are no other possibilities.

(b) [6 points] For each of the matrices M that you have found, determine its rank, image, and kernel.

2. Let A be the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}.$$

(a) [4 points] Find the inverse of A .

(b) [6 points] Find an invertible matrix S and a diagonal matrix D such that $D = S^{-1}AS$.

3. It is widely known that the Starship Enterprise is powered by Dilithium. Less well known is that of each 10 tons of Dilithium, only 7 remain at the end of a year; of the remaining 3 tons, one is converted to energy to run the warp drives etc, but the other 2 transmute to 2 tons of Trilithium. Trilithium is not stable either; in a year, of each 10 tons only 6 remain, with 1 ton converted to energy and the other 3 tons decaying back to Dilithium.

At Stardate 3000, the Starship's reactor is loaded with 1000 tons of Dilithium.

- (a) [2 points] Find a matrix A and an initial vector $\vec{x}(0)$ that encode this discrete linear dynamical system, that is such that at Stardate $3000 + k$ (k years after Stardate 3000), the amounts of Dilithium and Trilithium remaining in the reactor are the entries of the vector $\vec{x}(k) = A^k \vec{x}(0)$.
(You are **not** required to keep track of the energy released.)

- (b) [3 points] Find the characteristic polynomial and eigenvalues of A , and compute a basis of eigenvectors.

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- (c) [3 points] Compute a formula for the amounts of Dilithium and Trilithium remaining at Stardate $3000 + k$ as functions of k . What happens to these for large k ?
- (d) [2 points] Many years and Star Trek episodes later, it is found that only one ton of Dilithium is left in the reactor. Approximately how much Trilithium is there at the same time?

4. Let P be the plane in \mathbb{R}^3 spanned by the column vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

Let T be the orthogonal projection to P , that is $T = \text{proj}_P$.

(Note: you do **not** need to find the matrix for T to solve any of the following.)

(a) [3 points] Determine all the eigenvalues of T . For each eigenvalue, describe its eigenspace, and determine the eigenvalue's geometric multiplicity.

(b) [3 points] Find an **orthonormal** basis for \mathbb{R}^3 consisting of eigenvectors for T , and determine the corresponding eigenvalues.

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- (c) [4 points] Find the algebraic multiplicity of each eigenvalue of T . What is the characteristic polynomial of T ? Is T invertible? Why or why not?

5. Let B be the matrix

$$B = \begin{bmatrix} 2 & -4 & 0 \\ 5 & -6 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) [3 points] Find all eigenvalues of B .

(b) [3 points] Does the continuous dynamical system $\frac{d}{dt}\vec{x}(t) = B\vec{x}(t)$ have a point of stable equilibrium? Why or why not?

(c) [4 points] Describe qualitatively the behavior of this dynamical system if $\vec{x}(0)$ is the unit vector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

6. Recall that $C^\infty[0, 2]$ is the space of infinitely differentiable real functions on the interval $[0, 2]$. Given two functions f, g in $C^\infty[0, 2]$, their inner product is $\langle f, g \rangle = \int_0^2 f(x)g(x) dx$. Let P_1 be the subspace of $C^\infty[0, 2]$ consisting of polynomials of degree at most 1, that is, all functions of the form $a + bx$ where a and b are real numbers.

(a) [2 points] Show that P_1 is a linear subspace of $C^\infty[0, 2]$.

(b) [4 points] Find an orthonormal basis of P_1 .

(c) [4 points] Find the polynomial of degree at most 1 that best approximates (in the least-squares sense) the function $f(x) = x^3$ on the interval $[0, 2]$.

7. (a) [4 points] Find a basis of the kernel of the linear differential operator T defined by

$$T(f) = f'' + 8f' + 16f.$$

- (b) [2 points] Find an element in the kernel satisfying $f(0) = 1$ and $f'(0) = 2$.

- (c) [2 points] Find the general solution of the inhomogeneous differential equation

$$f'' + 8f' + 16f = 25e^t.$$

- (d) [2 points] Find the solution of $f'' + 8f' + 16f = 25e^t$ that satisfies the initial value conditions $f(0) = 2$ and $f'(0) = 2$.

8. Consider the temperature $T(x, t)$ of a metal bar extending from $x = 0$ to $x = \pi$. The temperature satisfies the heat equation

$$\frac{\partial T}{\partial t} = \mu \frac{\partial^2 T}{\partial x^2}$$

for some positive constant μ , and the ends are held at a constant temperature of zero, that is, $T(0, t) = 0$ and $T(\pi, t) = 0$ for all $t \geq 0$.

- (a) [3 points] Show that the functions $e^{-\mu n^2 t} \sin(nx)$ satisfy the differential equation and the initial conditions for all positive integers n .

- (b) [7 points] Suppose now that the initial temperature of the bar is given by $T(x, 0) = (\sin(x))^3$. Determine $T(x, t)$ for all $x \in [0, \pi]$ and all times $t \geq 0$. Describe the behavior of T as $t \rightarrow \infty$.
(Hint: Use Euler's formula $e^{iy} = \cos y + i \sin y$ if you are not sure about the relevant trigonometric identities.)

9. For each of the following 10 assertions, circle **T** if the assertion is **true**, and circle **F** if the assertion is **false**. Each is worth one point. For this question only, there is no need to justify your answers.

T F a) There exists a 2×2 matrix A such that each of A , $A + I_2$ and $A - I_2$ has rank 1.

T F b) If A is a 3×3 matrix whose characteristic polynomial is $1 - \lambda^3$, then $A^3 = I$.

T F c) There exists a plane P in \mathbb{R}^3 such that the projection of $2\vec{e}_1$ onto P is $\vec{e}_1 + \vec{e}_2 + \vec{e}_3$.

T F d) If w and z are complex numbers such that $e^w = e^z$, then $w = z$.

T F e) There is a continuous function f on $[-\pi, \pi]$ such that $\int_{-\pi}^{\pi} f(t) \sin(kt) dt = 1$ for each of $k = 1, 3, 5, 7, 9, \dots$

T F f) If A is a 2×2 matrix with eigenvalues of absolute value at most 1, then $\vec{0}$ is a stable equilibrium of the continuous dynamical system $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$.

T F g) If $f(x) = f(-x)$ for all x , then the Fourier series of the function $f(x)$ has no sine terms.

T F h) If A is an invertible symmetric matrix, then A^{-1} is also symmetric.

T F i) $T(f(x)) = x^3 f(x)$ is a linear map from C^∞ to C^∞ .

T F j) If A is a 2×2 matrix and the trace and determinant of A are both positive, then $\vec{0}$ is a stable equilibrium point of the discrete dynamical system $\vec{x}(t+1) = A\vec{x}(t)$.