MATH 21b Practice Questions

Problem 1.

Circle T if the given assertion is true, and circle F if it is false. There is no need to justify your answer.

T F a) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then AB = BA.

T F b) Suppose $\vec{u}$ is a non-zero vector in R^n. The map of R^n to itself that sends any given vector $\vec{v}$ to $T(\vec{v}) = \vec{v} + \vec{u}$ is a linear transformation.

T F c) For any square matrix A, the kernel of A^2 is a subspace of the kernel of A^3.

T F d) If $T: R^n \to R^m$ and $S: R^m \to R^n$ are linear transformations such that the kernel of S has dimension greater than zero, then the kernel of ST must have dimension greater than zero.

T F e) If a system of linear equations has more unknowns then equations, there are always an infinite number of solutions.

T F f) If the product of two matrices is 0, then one or the other must also be 0.

T F g) If A is a matrix, then ker(A) must be the same subspace as ker(rref(A)).

T F h) If A is a matrix, then image(A) must be the same as image(rref(A)).

T F i) If A is a square matrix with linearly independent columns, then the rows are also linearly independent vectors.

T F j) A linear transformation of R^2 cannot send $\begin{pmatrix} 6 \\ -9 \end{pmatrix}$ to $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ to $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Problem 2.

Let T denote a linear transformation of R^2 that sends $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

a) What are the coordinates of $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ with respect to the basis $\vec{v}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

b) Write down the matrix of T with respect to the basis $\vec{v}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

c) Write down the matrix of T with respect to the standard basis, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Problem 3.

Let A denote the matrix $\begin{pmatrix} 1 & 2 & 0 & 2 & 5 \\ 1 & 2 & 1 & -1 & 0 \\ 2 & 4 & -1 & 1 & 3 \\ 3 & 6 & -1 & -1 & 0 \end{pmatrix}$.

a) Compute rref(A).
b) Give a basis for kernel(A).
c) Give a basis for image(A).
d) What is the dimension of kernel(A^T)?

Problem 4.

Let A denote the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}$.

a) Give an orthonormal basis for the image of A.
b) Give the matrix T (with respect to the standard basis of R^3) that represents the orthogonal projection onto the image of A.
c) Let $\vec{v}_1$, $\vec{v}_2$ denote any orthonormal basis for image(A), and let $\vec{v}_3$ denote a unit length vector that is orthogonal to image(A). Find the matrix with respect to the basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ for the orthogonal projection to image(A).
d) Write a non-zero vector that is orthogonal to image(A).
e) Write down the matrix (with respect to the standard basis of R^3) that represents the orthogonal projection onto the orthogonal complement of image(A).

Problem 5.

Let T: $R^3 \rightarrow R^3$ denote the linear transformation whose matrix with respect to the standard basis of R^3 is $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 1 & 2 \end{pmatrix}$. Meanwhile, let $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

a) Compute A^2.
b) Prove that $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are a basis for R^3.
c) Write down the vector $A\vec{v}_1$.

Answers

1. a) F b) F c) T d) F e) F f) F g) T h) F i) T j) T .

2. a) $\vec{v}_1 + \vec{v}_2$ b) $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ c) $\begin{pmatrix} -2 & -1 \\ 7 & 3 \end{pmatrix}$.

3. a) $\begin{pmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ b) $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 0 \\ -1 \\ -2 \\ 1 \end{pmatrix}$ c) The last 3 columns of A span the image

   d) Since the kernel of A^T is the orthogonal complement to the image of A, and the image of A has dimension 3, the kernel of A^T has dimension 2.

4. a) $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

   b) $T = \frac{1}{6}\begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}$.

   c) The matrix is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

   d) $\vec{v} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

   e) The matrix is $\frac{1}{6}\begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}$

5\. a) $A^2 = \begin{pmatrix} 14 & 11 & 13 \\ 14 & 15 & 16 \\ 11 & 11 & 15 \end{pmatrix}$.

b) To prove that they are linearly independent, suppose $c_1\vec{v}_1 + c_2\vec{v}_2 + c_2\vec{v}_3 = 0$. The middle entry of vector on the left side is $c_2 + c_3$, so $c_2 = -c_3$. This means that the top entry of the vector on the left is c_1 and so $c_1 = 0$. If $c_1 = 0$ and if $c_2 = -c_3$, then the bottom entry on the left side is $3c_2 - c_2 = 2c_2$, so $c_2 = 0$. Thus $c_3 = 0$ also.

c) $A\vec{v}_1 = \begin{pmatrix} 7 \\ 6 \\ 7 \end{pmatrix}$