MATH 21b Practice Questions

Problem 1:

Circle **T** if the given assertion is <u>true</u>, and circle **F** if it is <u>false</u>. There is no need to justify your answer.

T F a) Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix}$. Then the equation $A\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ has no solution.

T F b) Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ in $\mathbb{R}^4$ are necessarily linearly independent if $\vec{v}_4$ is <u>not</u> equal to $\vec{v}_1+\vec{v}_2+\vec{v}_3$.

T F c) If A is an $n \times n$ matrix and A = AA, then A must be either 0 or the identity matrix.

T F d) Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 4 \\ -1 & -4 & -3 \end{pmatrix}$. Then $\text{rref}(A) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

T F e) If a matrix has kernel = $\{\vec{0}\}$, then it must be invertible.

T F f) The span of the rows of a matrix A must be the same as the span of the rows of rref(A).

T F g) If A is a 2×2 matrix and AAA is the identity, then A must be the identity matrix.

T F h) If A and B are $n \times n$ matrices and AB is invertible, then so are A and B.

T F i) A linear transformation of $\mathbb{R}^2$ that sends $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ must send $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 8 \end{pmatrix}$.

T F j) Let $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ denote a given basis for $\mathbb{R}^3$ and let $\vec{v} = -\vec{v}_1 - 3\vec{v}_3 + 2\vec{v}_2$. Meanwhile, let T denote the linear transformation of $\mathbb{R}^3$ with matrix $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 4 \\ -1 & -4 & -3 \end{pmatrix}$ with respect to the basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. Then $\vec{v}$ is in the image of T.

Problem 2: Let $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Meanwhile, let $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. This problem concerns the linear transformation, T: $\mathbb{R}^2 \to \mathbb{R}^2$ that sends $\vec{v}_1$ to $\vec{e}_1$ and $\vec{v}_2$ to $\vec{e}_2$.

a) Find the matrix of T with respect to the basis $(\vec{e}_1, \vec{e}_2)$.

b) Find a vector $\vec{u}$ such that $T\vec{u} = \vec{v}_1$.

c) Find the matrix of T with respect to the basis $(\vec{v}_1, \vec{v}_2)$.

d) Write down the matrix for T^{-1} with respect to the basis $(\vec{v}_1, \vec{v}_2)$.

Problem 3: This problem concerns $\vec{v}_1 = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$, $\vec{v}_4 = \begin{pmatrix} -5 \\ 0 \\ -1 \\ -2 \end{pmatrix}$, all vectors in $\mathbb{R}^4$.

a) Write down a basis for the span of these four vectors.

b) Let A denote the 4×4 matrix whose k'th column is $\vec{v}_k$. Here, $k \in \{1, 2, 3, 4\}$. Give a basis for the kernel of A.

c) Give a basis for the image of A.

Problem 4: Let $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}$.

a) Write down an orthonormal basis for the span of $\{\vec{v}_1, \vec{v}_2\}$.

b) Write down the matrix with respect to the standard basis of $\mathbb{R}^3$ for the linear transformation that gives the orthogonal projection to the span of $\{\vec{v}_1, \vec{v}_2\}$.

c) Write down the matrix with respect to the standard basis of $\mathbb{R}^3$ for the linear transformation that gives the orthogonal projection to the orthogonal complement of the span of $\{\vec{v}_1, \vec{v}_2\}$.

d) Give an orthogonal and invertible linear transformation of $\mathbb{R}^3$ that sends both of the vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ to the span of $\{\vec{v}_1, \vec{v}_2\}$.

Problem 5: This problem concerns the following five matrices:

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ -2 & 0 & 2 \end{pmatrix},$$

Note that these matrices differ slightly one from the other.

a) Compute AC

b) Compute $B + B^T$. Here, B^T denotes the transpose of B.

c) Which, if any, of the three matrices has columns that form a set $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ such that $\vec{a}_1$ is orthogonal to $\vec{a}_2$ and both $\vec{a}_1$ and $\vec{a}_2$ are orthogonal to $\vec{a}_3$?

d) Which, if any, of the three matricies can be written as r D where r is a real number and D is an orthogonal matrix?

e) Which, if any, of the three matrices is <u>not</u> invertible?

ANSWERS

Problem 1: a) **T** b) **F** c) **F** d) **T** e) **F** f) **T** g) **F** h) **T** i) **F** j) **T**.

Problem 2: a) $\frac{1}{3}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$ b) $\vec{u} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ c) $\frac{1}{3}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$ d) $\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$

Problem 3: a) $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ b) $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ c) $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

Problem 4: a) The vectors $\vec{u}_1 = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\vec{u}_2 = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$.

b) $A = \vec{u}_1 \vec{u}_1^T + \vec{u}_2 \vec{u}_2^T = \frac{1}{9}\begin{pmatrix} 5 & 4 & -2 \\ 4 & 5 & 2 \\ -2 & 2 & 8 \end{pmatrix}$.

c) $A = \frac{1}{9}\begin{pmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{pmatrix}$.

d) $A = \frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{pmatrix}$.

Problem 5: a) $\begin{pmatrix} 4 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 4 \end{pmatrix}$, b) = $\begin{pmatrix} 2 & 2 & -1 \\ 2 & -4 & 2 \\ -1 & 2 & -4 \end{pmatrix}$, c) A, d) None, e) B