# Math 21b: Practice questions for second midterm

1. Let $A$ be the matrix
$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix},$$

and let $\vec{b}_1, \vec{b}_2$ be the vectors
$$\vec{b}_1 = \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix}, \quad \vec{b}_2 = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}.$$

(a) Compute $\det(A)$. What does this tell you about solutions of the linear system $A\vec{x} = \vec{b}$ for arbitrary $\vec{b}$?

(b) Does $A\vec{x} = \vec{b}_1$ have an exact solution?

(c) Does $A\vec{x} = \vec{b}_2$ have an exact solution?

(d) Find an orthonormal basis for $\text{im}(A)$.

(e) Find a least-squares solution of $A\vec{x} = \vec{b}_1$.

(f) Find a least-squares solution of $A\vec{x} = \vec{b}_2$.

2. (a) [1 pt.] For a square matrix $A$, define the *characteristic polynomial* $f_A(\lambda)$, the *algebraic multiplicity* of an eigenvalue, and the *geometric multiplicity* of an eigenvalue.

(b) [2 pts.] For any scalar $a$ let $A$ be the matrix
$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & a & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Compute the characteristic polynomial $f_A(\lambda)$.

(c) [4 pts.] For which value(s) of $a$ does $A$ have an eigenvalue $\lambda$ of algebraic multiplicity at least 2? What is that $\lambda$ in each case?

(d) [2 pts.] For each of the cases you found in (c), what is the geometric multiplicity of $\lambda$?

(e) [1 pt.] For each of the cases you found in (c), is $A$ diagonalizable?

3. Let $A$ be the matrix $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$.

(a) Compute the eigenvalues of $A$.

(b) Find an eigenbasis for $A$.

(c) Find a matrix $S$ such that $D = S^{-1}AS$ is diagonal. What is $D$?

(d) Give a formula for $A^n$. Check your work by verifying that the cases $n = 1$ and $n = 2$ of your formula agree with $A$ and $A^2$.

True or False?

- If $\vec{x}$ and $\vec{y}$ are any vectors in $\mathbb{R}^n$ then $\|\vec{x} + \vec{y}\|^2 + \|\vec{x} - \vec{y}\|^2 = 2(\|\vec{x}\|^2 + \|\vec{y}\|^2)$.
- If $A$ is a matrix and $\vec{b}$ is a vector in $\ker A^T$ then $\vec{0}$ is a least-squares solution of $A\vec{x} = \vec{b}$.
- There exists a subspace $V$ of $\mathbb{R}^3$ such that $\dim V^\perp = \dim V$.
- If a matrix $A$ has $\operatorname{tr}(A) = 0$ then $A$ is not invertible.
- The matrix $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 5 & 6 & 7 & 0 \\ 8 & 9 & 10 & 11 \end{bmatrix}$ has positive determinant.
- An $n \times n$ matrix can have at most $n$ real eigenvalues.
- If $A$ is a square matrix such that $A^m = 0$ for some $m$ then $0$ is an eigenvalue of $A$.
- If $A$ is a square matrix such that $A^m = 0$ for some $m$ then $0$ is the only eigenvalue of $A$.