Math 21b
Fall 2006 Second Hourly
Solutions to Practice Problems 2 and 3
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Problem 2

(a) Fact 7.2.5: Characteristic polynomial. If A is an $n \times n$ matrix, then $\det(A - \lambda I_n)$ is a polynomial of degree n, of the form

$$(-\lambda)^n + (\operatorname{tr} A)(-\lambda)^{n-1} + \cdots + \det A.$$

This is called the *characteristic polynomial* of A, denoted by $f_A(\lambda)$. (Note that on the exam, it would be sufficient to say that $f_A(\lambda) = \det(A - \lambda I_n)$.)

Definition 7.2.6: Algebraic multiplicity of an eigenvalue. We say that an eigenvalue λ_0 of a square matrix A has *algebraic multiplicity* k if λ_0 is a root of multiplicity k of the characteristic polynomial $f_A(\lambda)$, meaning that we can write

$$f_A(\lambda) = (\lambda_0 - \lambda)^k g(\lambda)$$

for some polynomial $g(\lambda)$ with $g(\lambda_0) \neq 0$.

Definition 7.3.2: Geometric multiplicity. Consider an eigenvalue λ of an $n \times n$ matrix A. The dimension of eigenspace $E_\lambda = \ker(A - \lambda I_n)$ is called the *geometric multiplicity* of eigenvalue λ. Thus, the geometric multiplicity is the nullity of matrix $A - \lambda I_n$, or $n - \operatorname{rank}(A - \lambda I_n)$.

(b) $f_A(\lambda) = \det \begin{pmatrix} 1-\lambda & 1 & 1 \\ -1 & a-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda)[(1-\lambda)(a-\lambda)+1] = (1-\lambda)(\lambda^2 - (a+1)\lambda + a + 1).$

(c) We already know that 1 is an eigenvalue of A with algebraic multiplicity at least 1. But $\lambda - 1$ is not a factor of $g(\lambda) = \lambda^2 - (a+1)\lambda + a + 1$, because $g(1) = 1 \neq 0$.

Therefore, an eigenvalue of algebraic multiplicity 2 would have to be a double root of $g(\lambda)$. This happens when the discriminant is 0: $(a+1)^2 - 4(a+1) = (a+1)(a-3) = 0 \Rightarrow a = -1$ or $a = 3$. If $a = -1$, then $g(\lambda) = \lambda^2$, and $\lambda = 0$ is an eigenvalue with algebraic multiplicity 2. (Remember that we require *eigenvectors* to be nonzero, but it does make sense to have 0 as an *eigenvalue*.) If $a = 3$, then $g(\lambda) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$, and 2 is an eigenvalue with algebraic multiplicity 2.

(d) If $a = -1$, then $E_0 = \ker \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \operatorname{span} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$. Therefore 0 is an eigenvalue with geometric multiplicity 1.

If $a = 3$, then $E_2 = \ker \begin{pmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} = \operatorname{span} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Here 2 is an eigenvalue with geometric multiplicity 1.

(e) In neither case is A diagonalizable, since the geometric multiplicity of an eigenvalue is strictly less than its algebraic multiplicity.

Problem 3

(a) The characteristic polynomial of A is $f_A(\lambda) = \det(A - \lambda I_2) = \lambda^2 - (\text{tr } A)\lambda + \det A = \lambda^2 - 3\lambda - 10 = (\lambda + 2)(\lambda - 5)$. The eigenvalues are -2 and 5.

(b) $E_{-2} = \ker \begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix} = \text{span} \begin{pmatrix} 4 \\ -3 \end{pmatrix}$. $E_5 = \ker \begin{pmatrix} -4 & 4 \\ 3 & -3 \end{pmatrix} = \text{span} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(c) The change-of-basis matrix is $S = \begin{pmatrix} 4 & 1 \\ -3 & 1 \end{pmatrix}$. Then $D = S^{-1}AS$ should be the matrix with the eigenvalues of A on the diagonal, $\begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}$.

(d) $A = SDS^{-1}$, so $A^n = (SDS^{-1})^n = SD^nS^{-1} = \frac{1}{7} \begin{pmatrix} 4 & 1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} (-2)^n & 0 \\ 0 & 5^n \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 3 & 4 \end{pmatrix}$. We can multiply this out to obtain $\frac{1}{7} \begin{pmatrix} (-2)^{n+2} + 3 \cdot 5^n & -(-2)^{n+2} + 4 \cdot 5^n \\ -3(-2)^n + 3 \cdot 5^n & 3(-2)^n + 4 \cdot 5^n \end{pmatrix}$.

We can check our work by verifying that when $n = 1$, $SDS^{-1} = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} = A$, and when $n = 2$,

$SD^2S^{-1} = \begin{pmatrix} 13 & 12 \\ 9 & 16 \end{pmatrix} = A^2$.

A last reminder: A is similar to $B \Rightarrow f_A(\lambda) = f_B(\lambda) \Rightarrow \text{tr } A = \text{tr } B$ and $\det A = \det B$. The converses of these implications do not hold in general. Many of you were confused about this when doing the homework, so you should make sure you understand this before you take the exam.

If two matrices have the same characteristic polynomial, you should not assume that they are similar. For example, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ both have characteristic polynomial $f(\lambda) = \lambda^2$, but B has an eigenbasis while A does not (the eigenvalue 0 of A has algebraic multiplicity 2 but geometric multiplicity 1), so A and B are not similar.

Also, if $\det A = \det B$ and $\text{tr } A = \text{tr } B$, you should not assume that they have the same characteristic polynomial. The exception is when A and B are 2×2, in which case the characteristic polynomial $f_A(\lambda) = \lambda^2 - (\text{tr } A)\lambda + \det A$ is fully determined by the trace and determinant. For example, $A = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ both have trace 4 and determinant -3, but $f_A(\lambda) = -x^3 + 4x^2 - 3$ while $f_B(\lambda) = -x^3 + 4x^2 - 2x - 3$.