

**MATH 21b DIFFERENTIAL EQUATIONS**

Spring 2006

## 10. Differential equations

The goal of this the final part of the course is to introduce a great extension of the ideas from linear algebra to the realm of vector spaces with an infinite number of dimensions. I describe momentarily a hypothetical application to provide some inkling of the ‘raison d’etre’ for infinite dimensional linear algebra. This particular application concerns a differential equation; an equation for a function that involves constraints on its derivatives. Additional examples of differential equations appear in these notes.

Be forewarned that there are myriad scientific applications of infinite dimensional linear algebra; it is a tool worth knowing something about.

To start the story on the promised application, imagine a mercury thermometer of the kind that your folks might have used to take your temperature when you were a wee babe. The thermometer can be thought of as a long, thin bar of mercury, with most of it insulated. The part near one end is heated by inserting it under one’s tongue. Now imagine briefly heating this part of the thermometer and then removing the heat source. After such a momentary heating, the temperature will be different at different points along the thermometer, and these temperatures give the values of a function,  $\tau(x)$ , where  $x$  is a coordinate along the bar. Now I am left to wonder how the temperature at each point changes as a function of the time elapsed *after* the heat source is removed. I expect that the initial temperature disparities will decrease as time goes on. In any event, if I want to say something quantitative about the time and position dependence of the temperature, I am defacto searching for a function of the variables  $(t, x)$  that is defined for times  $t \geq 0$  and points  $x$  whose value at any given  $(t, x)$  tells me the temperature of the point  $x$  on the thermometer at time  $t$ . I call this sought after function  $T(t, x)$ . Here is a challenge: Find  $T(t, x)$  given the time  $t = 0$  temperature profile  $\tau(x)$ .

According to what was just said,  $T(0, x)$  is equal to  $\tau(x)$ . As argued in Section 10.4 below, this function  $T(t, x)$  is constrained to obey the *differential equation*

$$\frac{\partial}{\partial t} T = \mu \frac{\partial^2}{\partial x^2} T .$$

Here,  $\mu$  is a positive constant whose value is determined by various properties of the element mercury and by the units that are used to measure the position along the thermometer. As you can see, this equation says that the manner in which the temperature changes in time is determined by its  $x$  dependence at that time. In particular, it asserts that the derivative of  $T$  with respect to time is equal to  $\mu$  times the second derivative of  $T$  with respect to  $x$ . This particular differential equation has many names, one being the ‘heat equation’. My challenge to find  $T(t, x)$  amounts to solving the heat equation with the time 0 constraint  $T(0, x) = \tau(x)$ .

Where to begin? We do have experience in this course with solving equations for a time dependent vector in  $\mathbb{R}^n$ . I refer here to an equation such as

$$\frac{d}{dt} \vec{v} = A \vec{v}$$

where  $t \rightarrow \vec{v}(t)$  is the sought after vector function of time and  $A$  is a constant,  $n \times n$  matrix. Our techniques for solving this last sort of equation may be of some use to solving the heat equation provided that we justify the following analogy: The function  $T$  can play the role of the time dependent vector  $\vec{v}(t)$ , and the operation that sends

$$T \rightarrow \mu \frac{\partial^2}{\partial x^2} T$$

can play the role of *matrix* multiplication,  $\vec{v} \rightarrow A \vec{v}$ .

Humor me for a bit so that I can pursue this analogy. We know how to solve the vector equation when the matrix  $A$  is diagonalizable. Let me remind you how this is done: I first find a basis,  $\{\vec{u}_a\}_{a=1,2,\dots,n}$ , for  $\mathbb{R}^n$  where each basis vector is an eigenvector of the matrix  $A$ . This is to say that  $A \vec{u}_a = \lambda_a \vec{u}_a$  where  $\lambda_a$  is a real or complex number. I then write the time  $t = 0$  version of  $\vec{v}$  using this basis as

$$\vec{v}(0) = v_1 \vec{u}_1 + v_2 \vec{u}_2 + \dots + v_n \vec{u}_n .$$

The corresponding solution to the vector differential equation is

$$\vec{v}(t) = v_1 e^{\lambda_1 t} \vec{u}_1 + v_2 e^{\lambda_2 t} \vec{u}_2 + \dots + v_n e^{\lambda_n t} \vec{u}_n .$$

If I am to pursue this analogy to obtain our heat equation solution, then I must, perforce, obtain answers to the five questions that follow. Here is the first:

- *How can I view a function of  $x$  as a member of a vector space?*

If I can view functions of  $x$  in this way, then the assignment  $t \rightarrow T(t, \cdot)$  can be viewed as a ‘time dependent’ vector in this vector space of functions.

Here is the second question:

- *How can I view the operation of taking derivatives as that of a matrix or linear transformation acting on the vector space of functions?*

If I can view derivatives in this way, then I can view the assignment  $T \rightarrow \frac{\partial^2}{\partial x^2} T$  as the result of acting on a time dependent vector in my vector space of functions by a linear transformation.

What follows is the third question.

- *Granted that the assignment of  $\frac{d^2}{dx^2} f$  to a function  $f(x)$  can be interpreted as a linear transformation, what are its ‘eigenvectors’, the functions that obey  $\frac{d^2}{dx^2} f = \lambda f$  where  $\lambda$  is a real or complex number?*

If I can answer this third question, I am then faced with the fourth:

- *What is a basis for an infinite dimensional vector space? In particular, is there a basis whose elements obey  $\frac{d^2}{dx^2} f = \lambda f$  with  $\lambda$  a real or complex number?*

If there is such a basis, then I can solve the heat equation once I answer this last question:

- *How do I write the initial temperature profile  $\tau(x)$  as a linear combination of this basis of eigenvectors?*

When I come to terms with all of the above, then I can write down the desired function  $T(t, x)$ .

Answers to these five questions are part of the readings that follow. Take note, however, that the infinite dimensional linear algebra notions that are introduced in the ensuing discussion are used for much more than just the heat equation. They are tools that are wielded in all sorts of applications to the sciences.

### **10.1 Vector spaces whose elements are functions**

Many of the ideas of linear algebra which we have studied in the context of  $\mathbb{R}^n$  are applicable in a much wider context. Mathematicians introduced the abstract notion of a ‘vector space’, or what is a synonym, a ‘linear space’, to describe this greater context. Rather than look at linear spaces in the abstract, the discussion in this and the next two sections look specifically at examples that have applications to differential equations. The purpose of this section is to explain the following:

- *How to view the functions that can be differentiated any number of times as the elements of a vector space.*

- *The notions of subspace, linear dependence and linear independence for this vector space.*
- *How to view the act of taking a derivative as a linear transformation of this vector space.*
- *The kernel and image of a linear transformation that arises by taking derivatives.*
- *The notions of basis and dimension for a given subspace of this vector space.*

The discussion starts with the definition of what a mathematician refers to as a ‘smooth’ function: This is a function on the line,  $\mathbb{R}$ , that can be differentiated as often as desired. The set of all smooth functions is traditionally denoted as  $C^\infty$ . For example, the constant function  $f(t) = 1$  is a smooth function, as are  $g(t) = t$  and  $h(t) = e^t$ . These are all functions in the set  $C^\infty$ . To see that this is so, note that all derivatives of  $f$  vanish, all but the first of  $g$  vanish, and the  $n$ ’th derivative of  $h$  is equal to  $h$ . Thus, each function here can be differentiated as many times as desired. On the other hand,  $f(t) = |t|$  is not in  $C^\infty$  since it is not differentiable at  $t = 0$ .

The set  $C^\infty$  as an example of a ‘linear space’. To say that  $C^\infty$  is a ‘linear space’ means no more nor less than the following:

*If  $f$  and  $g$  are two functions in  $C^\infty$ , then so is the function  $t \rightarrow f(t) + g(t)$ ; and, if  $c$  is any real number and  $f$  is in  $C^\infty$ , then the function  $t \rightarrow c f(t)$  is also in  $C^\infty$ .*

Thus, one can add functions in  $C^\infty$  to get a new function in  $C^\infty$ , and one can multiply a function in  $C^\infty$  by a real number to get a new function in  $C^\infty$ . For example,  $1 \in C^\infty$  and  $\cos(3t) \in C^\infty$ , as is  $f(t) \equiv 1 + \cos(3t)$ . Likewise,  $t$  and also  $5t$  and  $-3.414 t$  are in  $C^\infty$ .

Addition of vectors and multiplication of vectors are the basic operations that we have studied on  $\mathbb{R}^n$ , and here we see a huge set,  $C^\infty$ , that admits these same two basic operations. In this regard, any set with these two operations, addition and multiplication by real (or complex) numbers, is what a mathematician means by a ‘vector space’ or, equivalently, a ‘linear space’.

Many of the same notions that we introduced in the context of vectors in  $\mathbb{R}^n$  have very precise counterparts in the context of our linear space  $C^\infty$ . What follows are some examples of particularly relevance to what we will do in the subsequent subsections.

- *Subspaces:* Any polynomial function,  $t \rightarrow a_n t^n + a_{n-1} t^{n-1} + \dots + a_0$  is infinitely differentiable, and so is in  $C^\infty$ . Here, each  $a_k$  is a real number. The set of all polynomials form a subset,  $P \subset C^\infty$ , with two important properties: First, if  $f(t)$  and  $g(t)$  are in  $P$ , then so is the function  $t \rightarrow f(t) + g(t)$ . Second, if  $c$  is a real number and  $f(t)$  is a polynomial, then the function  $t \rightarrow cf(t)$  is a polynomial. Thus, the sum of two elements in  $P$  is also in  $P$  and the product of a real number with an element of  $P$  is also an element of  $P$ .

If you recall, a subset  $V \subset \mathbb{R}^n$  of vectors was called a ‘subspace’ if it had the following two properties: Sums of vectors in  $V$  are in  $V$ , and any real number times any vector in  $V$  is  $V$ . It is for this reason that a set, such as  $P$ , is called a ‘subspace’ of  $C^\infty$ . Thus, a subspace is a subset that has the two salient properties of a vector space.

- *Linear independence:* Can you find two constants,  $c_1$  and  $c_2$ , that are not both zero and such that the function  $t \rightarrow c_1 + c_2t$  is zero for all values of  $t$ ? A moment’s reflection should tell you that if  $c_1 + c_2t$  is zero for all values of  $t$ , then  $c_1$  must be zero (try setting  $t = 0$ ), and also  $c_2$  must be zero (then set  $t = 1$ ).

If you recall, a set,  $\{v_1, \dots, v_k\}$ , of vectors in  $\mathbb{R}^n$  was said to be ‘linearly independent’ in the case that  $c_1 = c_2 = \dots = c_k = 0$  are the only values for a collection of constants  $\{c_1, \dots, c_k\}$  that makes  $c_1v_1 + c_2v_2 + \dots + c_kv_k = 0$ .

By the same token, functions  $\{f_1, \dots, f_k\} \subset C^\infty$  are said to be linearly independent in the case that  $c_1 = c_2 = \dots = c_k = 0$  are the only values for a collection of constants  $\{c_1, \dots, c_k\}$  that makes  $c_1f_1(t) + \dots + c_kf_k(t) = 0$  for all  $t$ . Note that I have underlined the notion that this sum is supposed to vanish for all choices of  $t$ , not just some choices. For example, the functions 1 and  $t$  are linearly independent, even though the function  $t \rightarrow 1+t$  is zero at  $t = -1$ .

To get a feeling for this notion, do you think that the functions  $\{1, t, \dots, t^n\}$  for any given non-negative integer  $n$  forms a linearly independent set? If you said yes, then you are correct. Here is why: Suppose  $t \rightarrow p(t) \equiv c_0 + c_1t + \dots + c_nt^n$  is zero for all  $t$  in the case that  $c_0, \dots, c_n$  are all constant. If such is the case, then  $p(0)$  must be zero, and so  $c_0$  is zero. Also, the derivative of  $p(t)$  must be the zero function (since  $p$  is) and this derivative is  $c_1 + 2c_2t + \dots + nc_nt^{n-1}$ . In particular,  $p'(0)$  must be zero and so  $c_1 = 0$  too. Continuing in this vein with the higher derivatives finds each successive  $c_k = 0$ .

A collection of functions that is not linearly independent is called ‘linearly dependent’.

- *Linear transformation:* If  $f(t) \in C^\infty$ , then we can define a new function that we will denote as  $Df$  by taking the derivative of  $f$ . Thus,

$$(Df)(t) = f'(t) = \frac{df}{dt}(t)$$

For example,  $D(\sin(t)) = \cos(t)$ .

Because we can take as many derivatives of  $f$  as we like, the function  $t \rightarrow (Df)(t)$  is a smooth function. Moreover,  $D$  has two important properties:

- a)  $D(f + g) = Df + Dg$  *no matter the choice for  $f$  and  $g$ .*

b)  $D(cf) = cDf$  if  $c$  is a constant.

If you recall, a transformation of  $\mathbb{R}^n$  with the analogous two properties was called a ‘linear transformation’. By analogy, we call  $D$  a linear transformation of  $C^\infty$ . Equivalently, we say that  $D$  is ‘linear’.

Here is another example: Set  $D^2f$  to denote the function  $t \rightarrow f''(t)$ . Then  $D^2$  is also linear. In general, so is  $D^n$  where  $D^n f$  takes the  $n$ 'th derivatives. Furthermore, so is the transformation that sends  $f$  to the function  $a_n D^n f + a_{n-1} D^{n-1} f + \dots + a_0 f$  in the case that the collection  $a_0, \dots, a_n$  are constants. In fact, such is the case even if each  $a_k$  is a fixed function of  $t$ . In this regard, be sure to define this transformation so that the same collection  $\{t \rightarrow a_k(t)\}$  is used as  $f$  varies in the set  $C^\infty$ .

- *The kernel of a linear transformation:* The kernel of a linear transformation such as  $D$  is the set of functions,  $f \in C^\infty$ , such that  $(Df)(t) = 0$  for all values of  $t$ . This is to say that  $Df$  is the zero function. In the case of  $D$ , a function has everywhere zero first derivative if and only if it is constant, so  $\ker(D)$  consists of the constant functions.

For a second example, consider  $D^2$ . What is the kernel of  $D^2$ ? Well a function whose second derivative is zero everywhere must have constant first derivative. Thus,  $D^2 f = 0$  if and only if  $f' = c_1$  with  $c_1$  a constant. But, a function with constant first derivative must have the form  $f = c_0 + c_1 t$ , where  $c_0$  is also constant. Arguing in this manner finds that the kernel of  $D^2$  consists of all functions of the form  $\{c_0 + c_1 t\}$  where  $c_0$  and  $c_1$  can be any pair of constants.

Note that the kernel of a linear transformation is always a linear subspace.

- *The image of a linear transformation:* A function, say  $t \rightarrow g(t)$ , is said to be in the ‘image’ of  $D$  if there is a smooth function  $f$  that obeys  $(Df)(t) = g(t)$  at all  $t$ . Thus,  $g$  is in the image of  $D$  if  $g$  has an anti-derivative that is a smooth function. Now, every function has an anti-derivative, and the anti-derivative is smooth if the original function is. To explain, if  $f' = g$  and I can take as many derivatives as I like of  $g$ , then I can take as many as I like of  $f$  and all are smooth. In particular, because  $f' = g$ , there is a first derivative. Moreover, any  $n \geq 1$  derivatives of  $f$  is  $n+1$  derivatives of  $g$ .

With the preceding understood, the image of  $D$  is the whole of  $C^\infty$ . Indeed, you give me any  $g(t)$  and I'll take the corresponding function  $f$  to be

$$t \rightarrow \int_0^t g(s) ds$$

By the way, I hope that it strikes you as passing strange that I have exhibited a linear transformation from  $C^\infty$  to itself whose image is the whole of  $C^\infty$  but whose kernel is non-zero. A little thought should convince you that a linear transformation of  $\mathbb{R}^n$  whose image is  $\mathbb{R}^n$  must have trivial kernel. This novel phenomena is a

manifestation of the fact that  $C^\infty$  is what is rightfully called an ‘infinite dimensional’ space. More is said on this below.

- *Basis, span and dimension:* As I argued previously, the kernel of  $D^2$  consists of all functions of the form  $c_0 + c_1 t$  where  $c_0$  and  $c_1$  are constants. Thus, the kernel of  $D^2$  consists of linear combinations of the two functions, 1 and  $t$ . These are then said to ‘span’ the kernel of  $D^2$  and as they are linearly independent, they are also said to give a ‘basis’ for the kernel of  $D^2$ . As this basis has two elements, so the kernel of  $D^2$  is said to be 2-dimensional.

As another example, the subspace,  $P_3$ , of polynomials of degree three or less consists of all functions of the form  $t \rightarrow c_0 + c_1 t + c_2 t^2 + c_3 t^3$  where each  $c_k$  is a constant. Now, the functions  $\{1, t, t^2, t^3\}$  are linearly independent, and they span  $P_3$  in the sense that every element in  $P_3$  is a linear combination from this set. Since there are four of the functions involved, the subspace  $P_3$  is said to be 4-dimensional.

In general, if  $V$  is a subspace,  $n \in \{0, 1, \dots\}$  and  $\{f_1, \dots, f_n\}$  is a set of linearly independent functions that span  $V$ , then  $V$  is said to be  $n$ -dimensional. To be precise here, a set  $\{f_1, \dots, f_n\}$  of functions in  $V$ , whether linearly independent or not, is said to span  $V$  if any given function  $g(t) \in V$  can be written as  $g(t) = c_1 f_1(t) + \dots + c_n f_n(t)$  where each  $c_k$  is a constant.

A subspace such as the space of all polynomials is said to be infinite dimensional if it has arbitrarily large subsets of linearly independent functions. It is in this sense that  $C^\infty$  itself is infinite dimensional.

Any linear operator on  $C^\infty$  that takes any given  $f(t)$  to some linear combination of it and its derivatives is an example of a ‘linear differential operator’. The general, form for a *linear differential operator* is a linear transformation,  $f \rightarrow Tf$ , of  $C^\infty$  that sends any given  $f(t)$  to the function

$$(Tf)(t) = a_n(t) (D^n f)(t) + a_{n-1}(t) (D^{n-1} f)(t) + \dots + a_0(t) f(t) ,$$

where each  $a_k$  is some smooth function. If each  $a_k$  is constant, then  $T$  is said to be a ‘constant coefficient’ differential operator. Granted that  $a_n \neq 0$ , then  $T$  is said to have ‘order  $n$ ’. For example,  $D^2$  is an such an example of order 2. Here is another:

$$(Tf)(t) = f''(t) + 3f'(t) - 2f(t) .$$

There is more arcane vocabulary to learn here. If  $T$  is a linear differential operator and if one is asked to *find the general solution to the homogeneous equation for  $T$* , then one is being asked to find the kernel of  $T$ , thus all functions  $f(t)$  such that  $(Tf)(t) = 0$  at every  $t$ . On the other hand, if  $g(t)$  is some given function and one is asked to solve

the *inhomogeneous equation*  $Tf = g$ , this means you should find all functions  $f$  such that  $(Tf)(t) = g(t)$ .

Here is an example: Suppose that you are asked to find all solutions to the inhomogeneous equation  $D^2f = e^t$ . You would answer: The general solution has the form  $f(t) = e^t + c_0 + c_1t$ , where  $c_0$  and  $c_1$  are constants.

By the way, this last example illustrates an important fact:

**Fact 10.1.1:** *If  $T$  is a given differential operator,  $g(t)$  a given function, and  $f_0$  some solution to the inhomogeneous equation  $Tf = g$ , then any other solution to this equation has the form  $f(t) = f_0(t) + h(t)$  where  $h$  is a function from the kernel of  $T$ . That is,  $Th = 0$ .*

This fact has a mundane proof: If  $f$  is also a solution, then  $T(f - f_0) = Tf - Tf_0 = g - g = 0$ , so it is necessarily the case that  $f - f_0$  is in the kernel of  $T$ . Even so, Fact 10.1.1 is quite useful, since it means that once you find the kernel of  $T$ , then you need only find a single inhomogeneous solution to know them all.

The task of finding an element in the kernel of a generic differential operator, or solving an associated inhomogeneous equation can be quite daunting. Often, there is no nice, algebraic expression for elements in the kernel, or for the solution to the inhomogeneous equation. Even so, there are some quite general ‘existence’ theorems that tell us when, and how many, solutions to expect. For example, consider the following:

**Fact 10.1.2:** *Suppose that  $T$  is a differential operator that has the form*

$$(Tf)(t) = \frac{d^n f}{dt^n}(t) + a_{n-1}(t) \frac{d^{n-1} f}{dt^{n-1}}(t) + \cdots + a_1(t) \frac{df}{dt}(t) + a_0(t)f(t)$$

*where  $a_0, \dots, a_{n-1}(t)$  are smooth functions. Then, the kernel of  $T$  has dimension  $n$ . Moreover, if  $g(t)$  is any given function, then there exists some  $f(t)$  such that  $Tf = g$ .*

Of course, this doesn’t tell us what the solution to the equation  $Tf = g$  looks like, but it does tell us that there is a solution whether or not we can find it explicitly.

Unfortunately, the proof of this fact takes us beyond where we can go in this course, so you will just have to take it on faith until you take a more advanced mathematics course.

### Exercises

1. Which of the following are subspaces of  $C^\infty$ ?
  - a) All continuous functions from  $\mathbb{R}$  to  $\mathbb{R}$ .

- b) All  $f \in C^\infty$  such that  $f(0) + f'(0) = 0$ .  
 c) All  $f \in C^\infty$  such that  $f + f' = 0$ .  
 d) All  $f \in C^\infty$  such that  $f(0) = 1$ .
2. Which of the following subsets of  $C^\infty$  consists of linearly independent functions?  
 a)  $1, t, t^2, t^3e^t$ .  
 b)  $1+t, 1-t, t^2, 1+t+t^2$ .  
 c)  $\sin(t), e^t, e^t\sin(t)$ .  
 d)  $\sin(t), \cos(t), \sin(t+\frac{\pi}{3})$ .
3. Which of the following maps are linear?  
 a)  $T: C^\infty \rightarrow \mathbb{R}$  given by  $T(f) = f(0)$ .  
 b)  $T: C^\infty \rightarrow C^\infty$  given by  $T(f) = f^2 + f'$ .  
 c)  $T: C^\infty \rightarrow \mathbb{R}^2$  given by  $T(f) = (f(0), f(1))$ .  
 d)  $T: C^\infty \rightarrow \mathbb{R}$  given by  $T(f) = \int_{0 \leq t \leq 1} f(t) dt$ .
4. Find a basis for the image of  $T: C^\infty \rightarrow C^\infty$  given by  $T(f) = f(0) + f'(0)t + (f(0)+f'(0))t^2$ .
5. Explain why the equation  $t f'(t) = 1$  has no solutions in  $C^\infty$ .
6. Let  $T(f) = t^2 f'(t) + 2t f(t)$ .  
 a) Suppose that  $T(f) = 0$ . If  $g(t) = t^2 f(t)$ , explain why  $g'(t) = 0$ .  
 b) Explain how to use the conclusions from a) to prove that  $\text{kernel}(T) = \{0\}$ .  
 c) Explain why the constant function 1 is not in the image of  $T$ .

## 10.2 Constant coefficient linear differential operators

Although it is no simple matter to write down the kernel of your generic differential operator, the situation is rather different if the operator has constant coefficients. In this case, the kernel can be found in a more or less explicit form; what follows in this subsection gives the details of the story. In particular, this subsection provides the following:

- *The formula for the general solution of the equation*

$$\frac{d^n f}{dt^n}(t) + a_{n-1} \frac{d^{n-1} f}{dt^{n-1}}(t) + \cdots + a_1 \frac{df}{dt}(t) + a_0 f(t) = 0.$$

*in the case that each  $a_k$  is constant.*

- *Instructions for finding those solutions to this same equation that obey extra constraints on the values of the function  $f$  and any of its derivatives at prescribed times.*

With the preceding understood, our first task is find all functions  $f(t)$  such that  $(Tf)(t) = 0$  in the case where

$$(Tf)(t) = \frac{d^n f}{dt^n}(t) + a_{n-1} \frac{d^{n-1} f}{dt^{n-1}}(t) + \cdots + a_1 \frac{df}{dt}(t) + a_0 f(t) .$$

Consider first the case where  $n = 1$  in which case we are looking for functions  $f(t)$  that obey the equation  $f' + a_0 f = 0$ . We can write this equation as

$$\frac{df}{f} = -a_0 dt,$$

and integrate both sides to find that  $\ln(f(t)) = -a_0 t + c$ , where  $c$  can be any constant. Thus, the general solution is

$$f(t) = b e^{-a_0 t} \text{ where } b \in \mathbb{R} .$$

You can see directly that the kernel is 1-dimensional as predicted by Fact 10.1.2. As described below, such exponential functions also play a key role in the  $n > 1$  cases.

To analyze the  $n > 1$  cases, let us recall Fact 7.5.2: The polynomial

$$\lambda \rightarrow p(\lambda) = \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_0$$

always factorizes as

$$p(\lambda) = (\lambda - \lambda_n) \cdots (\lambda - \lambda_1) ,$$

where each  $\lambda_k$  is a complex number. In this regard, keep in mind that a given complex number can appear as more than one  $\lambda_k$ . Also, keep in mind that if a given  $\lambda_k$  is complex, then its complex conjugate appears as some  $\lambda_j$  with  $j \neq k$ . In any event, the following summarizes the  $n > 1$  story:

**Fact 10.2.1:** *In the case that the numbers  $\{\lambda_1, \dots, \lambda_n\}$  are distinct, then the kernel of  $T$  consists of linear combinations with constant coefficients of the real and imaginary parts of the collection  $\{e^{\lambda_k t}\}_{1 \leq k \leq n}$ . To be more explicit, write each  $\lambda_k$  as  $\lambda_k = \alpha_k + i\beta_k$  with  $\alpha_k$  and*

$\beta_k$  real. In the case where the  $\{\lambda_k\}$  are distinct, the kernel of  $T$  is spanned by the functions in the set  $\{e^{\alpha_k t} \cos(\beta_k t), e^{\alpha_k t} \sin(\beta_k t)\}_{1 \leq k \leq n}$ . In the general case, introduce  $m_k$  to denote the number of times a given  $\lambda_k$  appears in the set  $\{\lambda_j\}_{1 \leq j \leq n}$ . Then the kernel of  $T$  is spanned by the collection  $\{p_k(t) e^{\alpha_k t} \cos(\beta_k t), p_k(t) e^{\alpha_k t} \sin(\beta_k t)\}$  where  $p_k(t)$  can be any polynomial of from zero up to  $m_k-1$ .

For example, consider the case where  $T = D^2 - 2D + 3$ . The resulting version of  $p(\lambda)$  is the polynomial  $\lambda^2 - 2\lambda + 3$ , and the latter factorizes as  $(\lambda+3)(\lambda-1)$ . According to Fact 10.2.1, the kernel of  $T$  is spanned by  $\{e^{-3t}, e^t\}$ . You can check yourself that both are in the kernel. They are also linearly independent. Indeed, you can see this because  $e^t$  gets very large as  $t \rightarrow \infty$  and  $e^{-3t}$  goes to zero as  $t \rightarrow \infty$ . Because Fact 10.1.2 tells us that the kernel is 2-dimensional, we therefore know that they must span the kernel also.

Another example is the case that  $T = D^2 + 1$ . The corresponding polynomial is the function  $p(\lambda) = \lambda^2 + 1$ . This one factorizes as  $(\lambda+i)(\lambda-i)$ . Thus, its roots are  $\pm i$ , and so Fact 10.1.3 asserts that the kernel is spanned by  $\{\cos(t), \sin(t)\}$ . Since the second derivative of  $\cos(t)$  is  $-\cos(t)$ , it is certainly the case that  $D^2 \cos(t) + \cos(t) = 0$ . Likewise,  $\sin(t)$  is in the kernel of  $D^2 + 1$  since the second derivative of  $\sin(t)$  is  $-\sin(t)$ . These are linearly independent as can be seen from the following argument: If  $c_1 \cos(t) + c_2 \sin(t) = 0$  for all  $t$  with  $c_1$  and  $c_2$  constant, then this is true at  $t = 0$ . But, at  $t = 0$ ,  $\cos(t) = 1$  and  $\sin(t) = 0$ , so  $c_1 = 0$ . but then  $c_2 = 0$  also. According to Fact 10.1.2, the dimension of the kernel of  $D^2 + 1$  is 2, so  $\{\cos(t), \sin(t)\}$  must span the kernel.

Here is a third example: Take  $T = D^3 - 3D^2 + 3D - 1$ . In this case, the corresponding polynomial  $p(\lambda)$  is  $(\lambda-1)^3$ . There is only one root here,  $\lambda = 1$ , and it appears with multiplicity 3. According to Fact 10.2.1, the kernel should be generated by the collection of functions  $\{e^t, t e^t, t^2 e^t\}$ . This is to say that every element in the kernel has the schematic form

$$f(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t = (c_1 + c_2 t + c_3 t^2) e^t$$

where  $c_1, c_2$  and  $c_3$  are constants. You are invited to take the prescribed derivatives to verify that  $(Tf)(t) = 0$  for all  $t$ . Even so, here is what might be an easier way to do this: First, exploit the factorizing of  $p(\lambda)$  as  $(\lambda-1)^3$  to audaciously write

$$Tf = (D-1)(D-1)(D-1)f$$

Now, note that  $(D-1)f = (c_2 + 2c_3) t e^t$ . This being the case, then  $(D-1)(D-1)f = 2c_3 e^t$ . Finally,  $(D-1)(D-1)(D-1)f = 2c_3(D-1)e^t$ , and this is zero because the derivative of  $e^t$  is  $e^t$ .

By the way, these three examples illustrate two important points, and also indicate how to prove Fact 10.2.1. These two points are discussed first, and then the proof of Fact 10.2.1 is sketched.

- *If  $\{\lambda_1, \lambda_2, \dots\}$  is any finite or infinite collection of real numbers with no two the same, then the functions in the corresponding collection*

$$\{e^{\lambda_1 t}, e^{\lambda_2 t}, \dots\}$$

*are linearly independent.*

This is to say that if  $\{c_1, c_2, \dots, c_k\}$  are any finite collection of constants and if

$$c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \dots + c_k e^{\lambda_k t} = 0$$

then  $c_1 = c_2 = \dots = c_k = 0$ . To prove that such is the case, just consider the largest number from the collection  $\{\lambda_1, \dots, \lambda_k\}$ . Call it  $\lambda$ . Then as  $t \rightarrow \infty$ , all of the terms in the sum of exponential function are very much smaller than  $e^{\lambda t}$ , and so its corresponding constant must be zero. This understood, go to the next largest number from  $\{\lambda_1, \dots, \lambda_k\}$  and make the same argument. Continue until sequentially until all  $\lambda$ 's are accounted for.

- *If  $\{\lambda_1 = \alpha_1 + i\beta_1, \lambda_2 = \alpha_2 + i\beta_2, \dots\}$  are any finite or infinite collection of complex numbers with no two the same, then the functions in the collection*

$$\{e^{\alpha_1 t} \cos(\beta_1 t), e^{\alpha_1 t} \sin(\beta_1 t), e^{\alpha_2 t} \cos(\beta_2 t), e^{\alpha_2 t} \sin(\beta_2 t), \dots\}$$

*are linearly independent in the sense that no linear combination of any finite subset from this collection will vanish at all  $t$  unless the constants involved are all zero.*

Indeed, the fact that functions with different  $\alpha$ 's are linearly independent is argued just as in the previous point, by looking at how they grow as  $t \rightarrow \infty$ . The argument in the general case is more involved and so will not be presented.

What follows are some remarks that are meant to indicate how to proceed with a rigorous proof of Fact 10.2.1 in the general case. To start the story, remember that the operator  $T$  is  $D^n + a_{n-1}D^{n-1} + \dots + a_0$ , and so determines a corresponding polynomial  $p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_0$ . Suppose that some real number,  $r$ , is a root of this polynomial. Thus,  $p(r) = 0$ . Since the derivative of  $e^{rt}$  is  $r e^{rt}$ , so  $D^k e^{rt} = r^k e^{rt}$  for any given non-negative integer  $k$ . As a consequence,  $T(e^{rt}) = r^n e^{rt} + a_{n-1}r^{n-1}e^{rt} + \dots + a_0 e^{rt} = p(r) e^{rt} = 0$  for

all  $t$ . Thus, we see that each real root of  $p(\lambda)$  determines a corresponding exponential function in the kernel of  $T$ .

Now suppose that  $\eta$  is a complex root of  $p(\lambda)$ . In this regard, remember that the complex conjugate  $\bar{\eta}$  is also a root of  $p$ . Also, recall from Section 9.2 that the derivative of the complex number valued function  $t \rightarrow e^{\eta t}$  is  $\eta e^{\eta t}$ . Thus,  $D^k e^{\eta t} = \eta^k e^{\eta t}$  for any given non-negative integer  $k$ . Now, write  $\eta = \alpha + i\beta$ , where  $\alpha$  and  $\beta$  are real, and remember that  $e^{\alpha t} \cos(\beta t) = \frac{1}{2}(e^{\eta t} + e^{\bar{\eta} t})$ . Thus, the  $k$ 'th derivative of  $e^{\alpha t} \cos(\beta t)$  is  $\frac{1}{2}(\eta^k e^{\eta t} + \bar{\eta}^k e^{\bar{\eta} t})$ . As a consequence,

$$T(e^{\alpha t} \cos(\beta t)) = \frac{1}{2}(p(\eta) e^{\eta t} + p(\bar{\eta}) e^{\bar{\eta} t}) = 0 \text{ for all values of } t.$$

Since  $e^{\alpha t} \sin(\beta t) = \frac{1}{2i}(e^{\eta t} - e^{\bar{\eta} t})$ , the same sort of argument proves that  $T(e^{\alpha t} \sin(\beta t)) = 0$  for all  $t$  as well. This then proves that every complex conjugate pairs  $\{\eta, \bar{\eta}\}$  of roots of  $p(\lambda)$  determines a corresponding pair,  $\{e^{\alpha t} \cos(\beta t), e^{\alpha t} \sin(\beta t)\}$  of linearly independent functions in the kernel of  $T$ .

Having digested the contents of the preceding two paragraphs, you are led inevitably to the conclusions of Fact 10.2.1 in the case that  $p(\lambda)$  has  $n$  distinct roots. Of course, this is predicated on your acceptance of the assertion in Fact 10.1.2 that the kernel of  $T$  is  $n$ -dimensional. It is also predicated on your acceptance of the conclusions in the second point three paragraphs back about linear independence.

The argument for the case when some real or complex number occurs more than once in the collection  $\{\lambda_1, \dots, \lambda_k\}$  is based on the following observation: The derivative of  $t^k e^{\eta t}$  is  $kt^{k-1}e^{\eta t} + \eta t^k e^{\eta t}$ . As a consequence,  $t^k e^{\eta t}$  is a solution to the inhomogeneous equation

$$(D-\eta)f = k t^{k-1} e^{\eta t} .$$

By the same token,

$$(D-\eta)(D-\eta)(t^k e^{\eta t}) = k(k-1) t^{k-2} e^{\eta t} .$$

Now, if we just iterate these last observations, we find that acting sequentially  $q$  times by  $(D-\lambda)$  on  $t^k e^{\eta t}$  gives

$$(D-\eta)^q(t^k e^{\eta t}) = k(k-1)\cdots(k-q+1) t^{k-q} e^{\eta t} \quad \text{if } q \leq k \quad \text{and} \quad (D-\eta)^q(t^k e^{\eta t}) = 0 \quad \text{if } q > k.$$

With the preceding in mind, suppose that some given real or complex number,  $\eta$ , is a root  $p(\lambda)$  that occurs some  $q$  times in the collection  $\{\lambda_1, \dots, \lambda_k\}$ . Let us renumber this list so that the last  $q$  of them are the ones that are equal to  $\eta$ . If we are willing to take the audacious step of *factorizing* the operator  $T$  by writing

$$T = (D-\lambda_1)\cdots(D-\lambda_{n-q})(D-\eta)^q,$$

we see that  $(Tf)(t) = 0$  if  $f(t)$  is any linear combination from the set  $\{e^{\eta t}, t e^{\eta t}, \dots, t^{q-1} e^{\eta t}\}$ . Indeed, this is because we have learned from the preceding paragraph that any such linear combination is already sent to zero by the factor  $(D-\eta)^q$ . As before, the real and complex parts of any such linear combination must also be sent to zero by  $T$ . Thus, since the real part of  $t^k e^{\eta t}$  is  $t^k e^{\alpha t} \cos(\beta t)$  and the imaginary part is  $t^k e^{\alpha t} \sin(\beta t)$ , we are led to Fact 10.2.1 for the cases when the collection of roots of  $p(\lambda)$  contains repeated values.

By the way, having just read the preceding two paragraphs, you now have every right to be nervous about ‘factorizing  $T$ ’ by manipulating  $D$  as if it were just a ‘number’ or a variable like  $\lambda$  rather than the much more subtle object that says ‘take the derivative of what ever is in front of me’. You will have to trust that this sort of outrageous move can be justified in a very rigorous way.

When using differential equation solutions to predict the future from present data, one can run into a problem of the following sort: Find all solutions to the differential equation  $D^n f + a_{n-1} D^{n-1} f + \cdots + a_0 f = 0$  where the value of  $f$  and certain of its derivatives are prescribed at fixed times. For example, find all solutions to  $D^2 f - 2Df + 2 = 0$  that obey  $f(0) = 1$  and  $f(\frac{\pi}{2}) = 2$ . This sort of problem is solved by first using Fact 10.2.1 to write the most general solution, and then searching for those that obey the given fixed time conditions. In the example just given, an appeal to Fact 10.2.1 finds that the general solution has the form

$$f(t) = a e^{-t} \cos(t) + b e^{-t} \sin(t)$$

where  $a$  and  $b$  can be any constants. This understood, then the condition  $f(0) = 1$  requires that  $a = 1$  but does not constrain  $b$  at all. Meanwhile, the condition that  $f(\frac{\pi}{2}) = 2$  demands that  $b = 2 e^{\pi/2}$ . Therefore, the solution to this particular constrained differential equation problem is  $f(t) = e^{-t} \cos(t) + 2e^{\pi/2} e^{-t} \sin(t)$ .

As second example using the same equation  $D^2 f - 2Df + 2 = 0$  asks for all solutions with  $f'(0) = 0$ . This condition reads  $-a + b = 0$ . Thus, all solutions to this constrained problem have the form  $f(t) = a e^{-t} (\cos(t) + \sin(t))$  where  $a$  is any constant.

With regards to these constrained problems: Conditions that are demanded on  $f$  or its derivatives at  $t = 0$  are usually called ‘initial conditions’.

### Exercises

1. Find a basis for the kernel of  $T: C^\infty \rightarrow C^\infty$  given by  $T(f) = f'' + f' - 12f$  and then find a smooth function that obeys the three conditions  $T(f) = 0$ ,  $f(0) = 0$  and  $f'(0) = 1$ .

2. Find a basis for the kernel of  $T: C^\infty \rightarrow C^\infty$  given by  $T(f) = f'' + 2f' + 2f$  and find a smooth function that obeys the three conditions  $T(f) = 0$ ,  $f(0) = 1$  and  $f(1) = 1$ .
3. Find a basis for the kernel of  $T: C^\infty \rightarrow C^\infty$  given by  $T(f) = f'' + 6f' + 9f$  and find a smooth function that obeys the three conditions  $T(f) = 0$ ,  $f'(0) = 1$  and  $f(1) = 0$ .
4. Assume that  $a$  is a constant and find a basis for the kernel of  $T(f) = f'' + a^2 f$ . For what values of the constant  $a$  are there non-zero elements in the kernel that are equal to zero at both  $t = 0$  and at  $t = \pi$ ?
5. Find a basis for the kernel of  $T: C^\infty \rightarrow C^\infty$  given by  $T(f) = f'' + f(0)$ . Note that  $T$  is a linear operator, but not of the sort that we have been discussing in this subsection.

### 10.3 Fourier series

In the preceding subsections, we looked at spaces of functions that behaved much like vectors in  $\mathbb{R}^n$ , but we did not look at any analogues of the concept of length, angle or dot product. In this section, I introduce these analogs and the notion of a ‘Fourier series’, an important example where they play a central role. In brief, this subsection comes to terms with the following facts:

- *The set of continuous functions on any given interval is a vector space.*
- *There are notions of ‘dot’ product and ‘length’ for elements in the vector space of continuous functions that are defined where  $t$  is between  $-\pi$  and  $\pi$ . In particular, a dot product between functions  $f$  and  $g$  is defined to equal  $\frac{1}{\pi} \int_{-\pi}^{\pi} f(t)g(t) dt$ ; the associated ‘length’ of a function  $f$  is defined to be the square root of  $\frac{1}{\pi} \int_{-\pi}^{\pi} f(t)^2 dt$ .*
- *The constant function  $\sqrt{\frac{1}{2}}$  together with the functions  $\{\cos(kt), \sin(kt)\}_{k=1,2,\dots}$  constitute an orthonormal basis with respect to this dot product for the vector space of continuous functions on the interval where  $-\pi \leq t \leq \pi$ .*
- *In particular, any given continuous function that is defined where  $-\pi \leq t \leq \pi$  can be written in a unique way as*

$$f(t) = a_0 \sqrt{\frac{1}{2}} + \sum_{k=1,2,\dots} (a_k \cos(kt) + b_k \sin(kt))$$

where each  $a_k$  and  $b_k$  are constants. Here,  $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt$  and for  $k \geq 1$ ,

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(kt) dt \quad \text{and} \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(kt) dt .$$

This representation of a function as a sum of sines and cosines is called a ‘Fourier series’.

Be forewarned that Fourier series are used throughout the sciences. In particular, you will almost surely see them used in any discussion of phenomena where periodic influences are suspected because the Fourier series decomposes a function as a sum of functions that are each periodic in the variable  $t$ .

In any event, to start the discussion on these points, recall that if  $a$  and  $b$  are real numbers with  $a < b$ , then  $[a, b]$  denotes the interval in  $\mathbb{R}$  of points  $t$  with  $a \leq t \leq b$ . Note that the end points of the interval are included.

Now introduce the notation  $C[-\pi, \pi]$  to denote the collection of all continuous functions from the interval  $[-\pi, \pi]$  to  $\mathbb{R}$ . For example,  $t$ ,  $\sin(t)$ ,  $|t|$ , and  $\frac{1}{t-4}$  are in  $C[-\pi, \pi]$ . The last of these illustrates the fact that we only care about the values of the function where  $t$  ranges from  $-\pi$  and  $\pi$ . Since  $4 > \pi$ , the fact that  $\frac{1}{t-4}$  blows up as  $t \rightarrow 4$  has no bearing on its appearance in the space  $C[-\pi, \pi]$ . On the other hand  $\frac{1}{t-2}$  is not in  $C[-\pi, \pi]$  since it is not defined at the point  $t = 2$  which is in the interval between  $-\pi$  and  $\pi$ . There

are also functions that are bounded and defined on all points where  $-\pi \leq t \leq \pi$  but are not in  $C[-\pi, \pi]$  because they lack the required continuity. Here is an example: The function  $f(t)$  that is defined to be 1 where  $t > 0$ , 0 at  $t = 0$  and  $-1$  where  $t < 0$ . The jump discontinuity of  $f$  as  $t$  crosses zero precludes its membership in  $C[-\pi, \pi]$ .

As with  $C^\infty$ , the collection  $C[-\pi, \pi]$  is a linear space. Indeed, if  $t \rightarrow f(t)$  and  $t \rightarrow g(t)$  are in  $C[-\pi, \pi]$ , then so is the function  $t \rightarrow f(t)+g(t)$  as is  $t \rightarrow r f(t)$  in the case that  $r$  is a real number.

I next define the analog of a dot product on  $C[-\pi, \pi]$ . For this purpose, let  $f(t)$  and  $g(t)$  be any two continuous functions that are defined where  $-\pi \leq t \leq \pi$ . Their dot product is then denoted by  $\langle f, g \rangle$ , a number that is computed by doing the integral

$$\langle f, g \rangle \equiv \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)g(t) dt .$$

I hope to convince you that this has the salient features of the usual dot product on  $\mathbb{R}^n$ . For example

- $\langle f, g \rangle = \langle g, f \rangle$ .
- *If  $r$  is a real number, then  $\langle r f, g \rangle = r \langle f, g \rangle$ .*
- *If  $f, g$  and  $h$  are any three functions in  $C[-\pi, \pi]$ , then  $\langle f + g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$ .*
- *If  $f$  is not the constant function 0, then  $\langle f, f \rangle > 0$ .*

I'll leave it to you to verify the first three. To verify the fourth, notice first that

$$\langle f, f \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)^2 dt .$$

Now, the function  $t \rightarrow f(t)^2$  is non-negative, so the integral for  $\langle f, f \rangle$  computes the area under the graph in the  $(t, y)$  plane of the function  $y = f(t)^2$ . As  $f(t)^2$  is non-zero at some point (since  $f$  is not the constant function 0), this graph rises above the axis at some point. Since  $f$  is continuous, it rises above nearly as much at nearby points as well. Thus, there is some area under the graph, so  $\langle f, f \rangle > 0$ .

For example, if you remember how to integrate  $t \sin(t)$ , you will find that the dot product between the functions  $t$  and  $\sin(t)$  is

$$\langle t, \sin(t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} t \sin(t) dt = 2.$$

(If you forgot how to integrate  $t \sin(t)$ , here is a hint: Think about integration by parts.) For another example, the dot product between the constant function 1 and the function  $\sin(t)$  is given by

$$\langle 1, \sin(t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(t) dt = \frac{1}{\pi} (-\cos(\pi) + \cos(-\pi)) = 0 .$$

By analogy with the case of vectors in  $\mathbb{R}^n$ , I say that a pair of functions  $f$  and  $g$  from  $C[-\pi, \pi]$  are ‘orthogonal’ in the case that  $\langle f, g \rangle = 0$ . Thus, 1 and  $\sin(t)$  are orthogonal, but  $t$  and  $\sin(t)$  are not.

Just as we defined the length of a vector in  $\mathbb{R}^n$  using the dot product, so I define the length of any given function  $f \in C[-\pi, \pi]$  to be

$$\sqrt{\langle f, f \rangle}$$

The length of  $f$  is denoted here and elsewhere as  $\| f \|$ , and this number is called the ‘norm’ of  $f$ . By analogy with the case of  $\mathbb{R}^n$ , I define the distance between functions  $f$  and  $g$  from  $C[-\pi, \pi]$  to be

$$\| f - g \| = \sqrt{\langle f - g, f - g \rangle}$$

This is to say that the square of the distance between  $f$  and  $g$  is equal to

$$\frac{1}{\pi} \int_{-\pi}^{\pi} (f(t)-g(t))^2 dt .$$

According to this definition of distance,  $f$  is close to  $g$  in the case that  $f(t)$  is close to  $g(t)$  for all  $t \in [-\pi, \pi]$ . However, be forewarned that this definition doesn’t require that  $f(t)$  be close to  $g(t)$  at every  $t$ ; only that they be suitably close for ‘most’ values of  $t$ . You will see this in the third and fourth examples below.

Here are some examples of norms and distances:

- The constant function 1 has norm  $\| 1 \| = \sqrt{2}$  since this is the square root of  $\frac{1}{\pi}$  times the length of  $[-\pi, \pi]$ .
- The the square of the norm of the function  $t$  is

$$\langle t, t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} t^2 dt = \frac{2}{3} \pi^2 .$$

Thus, the norm of  $t$  is  $\| t \| = \sqrt{\frac{2}{3}} \pi$ .

- Let  $R$  be a positive real number. Then the distance between the function  $f(t) = t$  and the function  $g(t) = t + e^{-R|t|}$  is the square root of

$$\frac{1}{\pi} \int_{-\pi}^{\pi} e^{-2R|t|} dt = \frac{1}{\pi} \frac{1}{R} (1 - e^{-2R\pi}) .$$

Note in particular that the larger the value of  $R$ , the smaller the distance and as  $R \rightarrow \infty$ , the distance in question limits to zero. Even so,  $|f(0) - g(0)| = 1$  no matter how large  $R$ .

- Let  $R$  be a positive real number. Then the distance between the function  $f(t) = t$  and the function  $g(t) = t + R^{1/4} e^{-R|t|}$  is the square root of

$$\frac{1}{\pi} \frac{1}{\sqrt{R}} (1 - e^{-2R\pi}).$$

Note that in this variation of the previous example, the distance between  $f$  and  $g$  again limits to zero as  $R \rightarrow \infty$ , even though  $|f(0) - g(0)| = R^{1/4}$  now *blows up* as  $R \rightarrow \infty$ . The point here and in the previous example is that two functions in  $C[-\pi, \pi]$  can be close and still have widely different values at some  $t$ . As remarked previously, their values need only be suitably close at most  $t \in [-\pi, \pi]$ . I can't criticize you for thinking that this phenomena illustrates a serious defect in our notion of distance. The fact is that for some uses, other notions of distance are necessary for this very reason.

Granted this dot product for the linear space  $C[-\pi, \pi]$ , I now introduce the notion of an 'orthonormal' set of functions. This notion is the analog of the notion of orthonormality that we used for vectors in  $\mathbb{R}^n$ . In particular, a finite or infinite collection  $\{f_1, f_2, \dots\}$  of functions is deemed 'orthonormal' in the case that

$$\|f_k\| = 1 \text{ for all } k \quad \text{and} \quad \langle f_j, f_k \rangle = 0 \text{ for all unequal } j \text{ and } k.$$

For example, the constant function  $\sqrt{\frac{1}{2}}$  and the function  $\frac{1}{\pi} \sqrt{\frac{3}{2}} t$  comprise a two element orthonormal set. Indeed, the computations done previously for the norms of 1 and  $t$  justify the assertion that these two functions both have norm 1. Meanwhile, to see that these two functions are orthogonal, first note that the dot product between 1 and  $t$  is  $\frac{1}{\pi}$  times the integral of  $t$  from  $-\pi$  to  $\pi$ . Then note that the latter integral is zero since it is the difference between the values of  $\frac{1}{2\pi} t^2$  at  $t = \pi$  and  $t = -\pi$ .

Here is another example: The functions in the set

$$\left\{ \sqrt{\frac{1}{2}}, \frac{1}{\pi} \sqrt{\frac{3}{2}} t, \frac{1}{\pi^2} \frac{3\sqrt{5}}{4} (t^2 - \frac{1}{3}\pi^2) \right\}$$

is also orthonormal.

You most probably will recognize the following facts as  $C[-\pi, \pi]$  analogs of assertions that hold for vectors in  $\mathbb{R}^n$ :

- *If  $\{f_1, f_2, \dots, f_N\}$  is an orthonormal set, then they are linearly independent and so form a basis for their span.*
- *If  $h$  and  $g$  are orthogonal functions in  $C[-\pi, \pi]$ , then  $\|h \pm g\|^2 = \|h\|^2 + \|g\|^2$ .*

- Suppose that  $V$  is a subspace of  $C[-\pi, \pi]$  and that  $f \in C[-\pi, \pi]$ . If  $g$  is in  $V$  and if the function  $f - g$  is orthogonal to all functions in  $V$ , then  $\|f - g\| \leq \|f - h\|$  if  $h$  is in  $V$ . Moreover, this inequality is an equality only in the case that  $h = g$ .
- If  $\{f_1, \dots, f_N\}$  is an orthonormal basis for a subspace  $V \subset C[-\pi, \pi]$  and if  $f$  is any function in  $C[-\pi, \pi]$ , then the function in  $V$  we call  $\text{proj}_V(f)$  that is given by

$$\text{proj}_V(f)(t) = \langle f, f_1 \rangle f_1(t) + \dots + \langle f, f_N \rangle f_N(t)$$

is the closest function in  $V$  to  $f$ . Thus,  $f - \text{proj}_V f$  is orthogonal to each element in  $V$ .

- If  $V \subset C[-\pi, \pi]$  is a finite dimensional subspace, then  $V$  has an orthonormal basis.

The arguments for these last facts are essentially identical to those that prove the  $\mathbb{R}^n$  analogs. For example, to prove the first point, assume that  $g(t) \equiv c_1 f_1(t) + \dots + c_N f_N(t)$  is zero for all  $t \in [-\pi, \pi]$  where  $c_1, \dots, c_N$  are constants. Now take the dot product of  $g$  with  $f_1$  to find  $0 = \langle f_1, g \rangle = c_1 \langle f_1, f_1 \rangle + c_2 \langle f_1, f_2 \rangle + \dots + c_N \langle f_1, f_N \rangle$ . Because of the orthonormality, this equality boils down to  $0 = c_1 \cdot 1 + c_2 \cdot 0 + \dots + c_N \cdot 0$ , so  $c_1 = 0$ . Take the dot product of  $g$  with  $f_2$  to find that  $c_2$  is zero, then  $f_3$ , etc.

As a second example, here is how to prove the final point: The first thing to note is that it suffices to prove that  $f - \text{proj}_V f$  is orthogonal to every function in  $V$ . Indeed, if this is the case, then the version of the second point above with  $h = f - \text{proj}_V f$  and  $g$  any function in  $V$  proves that  $\text{proj}_V f$  is the closest function in  $V$  to  $f$  if  $f - \text{proj}_V f$  is orthogonal to every function in  $V$ . In any event,  $f - \text{proj}_V f$  is orthogonal to every function in  $V$  if and only if it is orthogonal to every basis function, that is each of  $f_1, \dots, f_N$ . Computing the dot product of  $f$  with any given  $f_k$  finds  $\langle f_k, f \rangle$ , and this is precisely the same as the dot product of  $f_k$  with  $\text{proj}_V f$ . Thus, the dot product of any given  $f_k$  with  $f - \text{proj}_V f$  is zero.

With regards to the final point, you won't be surprised to learn that the Gram-Schmid algorithm that we used in the case of  $\mathbb{R}^n$  to find an orthormal basis works just fine in the case of  $C[-\pi, \pi]$ . For example, the linear span of the functions 1 and  $t^2$  is a 2-dimensional subspace of  $C[-\pi, \pi]$ . Indeed, if  $c_1 + c_2 t^2$  is zero for all  $t$  with  $c_1$  and  $c_2$  constant, then it is zero at  $t = 0$  and so  $c_1 = 0$ . It is also zero at  $t = 1$ , and so  $c_2 = 0$  as well. To find an orthonormal basis, I first divide the constant function 1 by its norm to get a function with norm 1. The latter is  $\sqrt{\frac{1}{2}}$ . Next, I note that  $t^2 - \langle \sqrt{\frac{1}{2}}, t^2 \rangle \sqrt{\frac{1}{2}} = t^2 - \frac{1}{3}\pi^2$  is orthogonal to  $\sqrt{\frac{1}{2}}$ . Thus, I get an orthonormal basis for the span of  $\{1, t^2\}$  by using  $\sqrt{\frac{1}{2}}$  as the first basis element, and using for the second the function that you get by dividing the function  $t^2 - \frac{1}{3}\pi^2$  by the square root of the integral from  $-\pi$  to  $\pi$  of  $(t^2 - \frac{1}{3}\pi^2)^2$ .

Left unsaid in the final point above is whether any given infinite dimensional subspace of  $C[-\pi, \pi]$  has an orthonormal basis. The answer depends to some extent on how this question is interpreted. In any event, the next fact asserts that  $C[-\pi, \pi]$  itself has an infinite orthonormal basis. Moreover this basis 'spans'  $C[-\pi, \pi]$  in a certain sense that

is explained below. The fact is that  $C[-\pi, \pi]$  has many such basis, but only the most commonly used one is presented below.

**Fact 10.3.1:** *The collection  $\{ \sqrt{\frac{1}{2}}, \cos(t), \sin(t), \cos(2t), \sin(2t), \cos(3t), \sin(3t), \dots \}$  is an orthonormal set of functions in  $C[-\pi, \pi]$ .*

This fact is proved by verifying that the following integrals have the asserted values:

- $\langle \sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}} \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} dt = 1.$
- $\langle \sqrt{\frac{1}{2}}, \cos(nt) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sqrt{\frac{1}{2}} \cos(nt) dt = 0$  for any  $n \geq 1.$
- $\langle \sqrt{\frac{1}{2}}, \sin(nt) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sqrt{\frac{1}{2}} \sin(nt) dt = 0$  for any  $n \geq 1.$
- $\langle \cos(nt), \cos(nt) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2(nt) dt = 1$  for any  $n \geq 1.$
- $\langle \sin(nt), \sin(nt) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2(nt) dt = 1$  for any  $n \geq 1.$
- $\langle \cos(nt), \sin(mt) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nt) \sin(mt) dt = 0$  for any  $n$  and  $m.$
- $\langle \cos(nt), \cos(mt) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nt) \cos(mt) dt = 0$  for any  $n \neq m \geq 1.$
- $\langle \sin(nt), \sin(mt) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(nt) \sin(mt) dt = 0$  for any  $n \neq m \geq 1.$

To explain the sense in which the basis in Fact 10.3.1 spans  $C[-\pi, \pi]$ , let me introduce, for each positive integer  $N$ , the subspace  $T_N \subset C[-\pi, \pi]$  that is given by the span of

$$\left\{ \sqrt{\frac{1}{2}}, \cos(t), \sin(t), \dots, \cos(Nt), \sin(Nt) \right\}.$$

If  $f$  is any given function in  $C[-\pi, \pi]$ , I can take the projection of  $f$  onto  $T_N$ . This is the function

$$\text{proj}_{T_N} f \equiv a_0 \sqrt{\frac{1}{2}} + a_1 \cos(t) + b_1 \sin(t) + \dots + a_N \cos(Nt) + b_N \sin(Nt),$$

where

$$a_0 = \sqrt{\frac{1}{2}} \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt, \quad a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(kt) f(t) dt, \quad \text{and} \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(kt) f(t) dt.$$

With this notation set, here is what I mean by ‘span’:

**Fact 10.3.2:** *Let  $f$  be any function in  $C[-\pi, \pi]$ . Then  $\lim_{N \rightarrow \infty} \| f - \text{proj}_{T_N} f \| = 0.$*

Moreover, if the derivative of  $f$  is defined and continuous, then  $\lim_{N \rightarrow \infty} (\text{proj}_{T_N} f)(t) = f(t)$  if  $t$  lies strictly between  $\pi$  and  $-\pi$ . This assertion also holds at  $t = \pi$  and at  $t = -\pi$  in the case that  $f(\pi) = f(-\pi)$ . In any event, whether  $f$  is or is not differentiable, the infinite series  $a_0^2 + a_1^2 + b_1^2 + \cdots + a_k^2 + b_k^2 + \cdots$  is convergent and its limit is  $\|f\|^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)^2 dt$ .

By virtue of Fact 10.3.2, one often sees a given function  $f \in C[-\pi, \pi]$  written as

$$f(t) = a_0 \sqrt{\frac{1}{2}} + \sum_{k=1}^{\infty} (a_k \cos(kt) + b_k \sin(kt)),$$

where the collection  $\{a_k, b_k\}$  are given just prior to Fact 10.3.2. As noted at the start of this subsection, such a representation of  $f$  is called its ‘Fourier series’. The name honors the mathematician who first introduced these series, Jean-Baptiste-Joseph Fourier. Fourier was born in 1768 and lived until 1830.

The Fourier series for a given function  $f$  exhibits  $f$  as a sum of trigonometric functions and Fact 10.3.2 asserts the rather remarkable claim that every continuous function on the interval  $[-\pi, \pi]$  can be suitably approximated by such a sum.

The proof of Fact 10.3.2 is subtle and, but for the next remark, goes beyond what we will cover in this course. If the series  $a_0 \sqrt{\frac{1}{2}} + a_1 \cos(t) + b_1 \sin(t) + \cdots$  is convergent at each  $t$  with limit  $f(t)$ , then the convergence of the infinite series  $a_0^2 + a_1^2 + b_1^2 + \cdots$  is an automatic consequence of the fact that the collection  $\{\sqrt{\frac{1}{2}}, \cos(t), \sin(t), \cdots\}$  is an orthonormal set of functions. To see why, take some large integer  $N$  and write

$$f = \text{proj}_{T_N} f + (f - \text{proj}_{T_N} f).$$

Now, as discussed earlier, the two terms on the right hand side of this equation are orthogonal. This then means that

$$\|f\|^2 = \|\text{proj}_{T_N} f\|^2 + \|f - \text{proj}_{T_N} f\|^2.$$

By virtue of the fact that  $\{\sqrt{\frac{1}{2}}, \cos(t), \sin(t), \cdots\}$  is orthonormal, the first term on the right hand side of this last equation is  $a_0^2 + a_1^2 + b_1^2 + \cdots + a_N^2 + b_N^2$ . As a consequence, we see that

$$\|f\|^2 = a_0^2 + a_1^2 + b_1^2 + \cdots + a_N^2 + b_N^2 + \|f - \text{proj}_{T_N} f\|^2.$$

Thus, under the assumption that the limit as  $N \rightarrow \infty$  of the far right term above is zero, we then have our derivation of the asserted limit for the infinite sum  $a_0^2 + a_1^2 + b_1^2 + \cdots$ .

Here are some examples of Fourier series that give well known functions:

- $t = 2 \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} \sin(kt) .$
- $t^2 = \frac{1}{3} \pi^2 + 4 \sum_{k=1}^{\infty} (-1)^k \frac{1}{k^2} \cos(kt) .$
- $e^t = \frac{1}{\pi} (e^{\pi} - e^{-\pi}) \left[ \frac{1}{2} + \sum_{k=1}^{\infty} (-1)^k \left( \frac{1}{1+k^2} \cos(kt) - \frac{k}{1+k^2} \sin(kt) \right) \right].$

As you can see, the Fourier series of some very simple functions have infinitely many terms.

Here is how to see what this is all about: Get hold of a graphing calculator, and do the following: First graph the function  $y = t$  with the range  $-\pi \leq t \leq \pi$  and save it to compare with the next series of graphs. Since the Fourier series of  $t$  starts with the function  $2 \sin(t)$ , make the next graph that of  $y = 2 \sin(t)$ . As the first two terms in the Fourier series for  $t$  are  $2 \sin(t) - \sin(2t)$ , the third graph is that of  $y = 2 \sin(t) - \sin(2t)$ . Continuing in this vein, successively draw the graphs of  $y = 2 \sin(t) - \sin(2t) + \frac{2}{3} \sin(3t)$ ; then  $y = 2 \sin(t) - \sin(2t) + \frac{2}{3} \sin(3t) - \frac{1}{2} \sin(4t)$ ; then the graph of  $y = 2 \sin(t) - \sin(2t) + \frac{2}{3} \sin(3t) - \frac{1}{2} \sin(4t) + \frac{2}{5} \sin(5t)$ ; and so on. If you do this, you will see something quite amazing: The successive graphs come closer and closer to the original graph of  $y = t$ .

You should make a similar sequence of graphs for the cases  $y = t^2$  and  $y = e^t$  using the first few terms of their Fourier series. If you draw these graphs, you will get a good indication of how the Fourier series sum can converge to a given function.

When looking at the Fourier series given for the function  $t$ , what do you make of the fact that  $\pi$  is definitely not zero, but  $\sin(k\pi)$  is zero for all  $k$ ? In particular, the asserted ‘equality’ between the right and left hand sides in the first example is definitive nonsense at  $t = \pi$ . Even so, this does not violate the assertion of Fact 10.3.2 because the function  $t$  obviously does not have the same value at  $\pi$  as it does at  $-\pi$ . With regards to Fact 10.3.2, the equality in the first example holds only in the following sense:

$$\lim_{N \rightarrow \infty} \frac{1}{\pi} \int_{-\pi}^{\pi} (t - 2\pi \sum_{k=1}^N (-1)^{k+1} \frac{1}{k} \sin(kt))^2 dt = 0.$$

Thus, the equality in the first point holds at ‘most’ values of  $t$  in  $[-\pi, \pi]$ , but not at all values of  $t$ .

Contrast this with the equality between  $t$  and its Fourier series at  $\frac{\pi}{4}$ . According to Fact 10.3.2, the equality does indeed hold here, and so we obtain the following remarkable equality:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \dots .$$

As a professional skeptic, I checked this last equation on my calculator and here is what I found: First,  $\frac{\pi}{4} \sim 0.7854$ . By comparison, the first two terms on the right side above are  $1 - \frac{1}{3} \sim 0.667$ ; the first three terms are  $1 - \frac{1}{3} + \frac{1}{5} \sim 0.8667$ ; the first four give  $0.7238$ ; the

first five 0.8349; the first ten 0.7605, the first twenty 0.7729; the first one hundred 0.7829; and the first one thousand 0.7851. Thus the sum on the right side of this last equation does appear to approach  $\frac{\pi}{4}$ .

Other fantastic sums can be had by evaluating the right hand side of the equality between  $t^2$  and its Fourier series at some special cases. For example, the respective  $t = 0$  and  $t = \pi$  cases yield

$$\frac{1}{12}\pi^2 = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \dots \quad \text{and} \quad \frac{1}{6}\pi^2 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots .$$

By the way, the second of these equalities is equivalent to the assertion in Fact 10.3.2 that the value,  $\frac{2}{3}\pi^2$ , of  $\|t\|^2$  is equal to the sum of the squares of the coefficients that appear in front of the various factors of  $\sin(kt)$  in the Fourier series expansion given above for  $t$ .

Here are the key notions to remember from 10.3:

- The space  $C[-\pi, \pi]$  has an dot product whereby the dot product of any given two functions  $f$  and  $g$  is equal to  $\frac{1}{\pi} \int_{-\pi}^{\pi} f(t)g(t) dt$ . This is denoted by  $\langle f, g \rangle$ .
- The norm of a function  $f$  is  $\langle f, f \rangle^{1/2}$ , it is positive unless  $f$  is the constant function 0.
- The distance between any two given functions  $f$  and  $g$  is the norm of  $f - g$ .
- Most constructions in  $\mathbb{R}^n$  that use the dot product work as well here. In particular, any finite dimensional subspace has an orthonormal basis, and one can use this basis to define the projection onto the subspace.
- There is an orthonormal basis for  $C[-\pi, \pi]$  that consists of the constant function  $\sqrt{\frac{1}{2}}$  plus the collection  $\{\cos(kt), \sin(kt)\}_{k=1,2,\dots}$ . Any given function  $f$  can be depicted using this basis as

$$f(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(kt) + b_k \sin(kt)),$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt, \quad a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(kt) f(t) dt \quad \text{and} \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(kt) f(t) dt.$$

- The convergence of the series above to  $f(t)$  might not occur at all values of  $t$ , but in any event, the integral from  $-\pi$  to  $\pi$  of the square of the difference between  $f$  and the series truncated after  $N$  terms tends to zero as  $N$  tends to infinity.

## Exercises

1. Find an orthonormal basis for the subspace of  $C[-\pi, \pi]$  spanned by  $\{1, e^t, e^{-t}\}$  and then compute the projection of the function  $t$  onto this subspace.
2. Find the Fourier series for the function  $|t|$  on the interval  $[-\pi, \pi]$ .
3. If  $a$  is a real constant, find the Fourier series for  $\cosh(at)$  on the interval  $[-\pi, \pi]$  and use the result to derive a closed form formula for  $\sum_{n=1}^{\infty} \frac{1}{n^2+a^2}$ .
4. Let  $r \in \mathbb{R}$ . Prove that the collection  $\sqrt{\frac{1}{2}}, \{\cos(k(t-r)), \sin(k(t-r))\}_{k=1, \dots}$  is an orthonormal basis for  $C[-\pi+r, \pi+r]$  using the dot product that assigns any two given functions  $f$  and  $g$  the number  $\frac{1}{\pi} \int_{-\pi+r}^{\pi+r} f(t)g(t) dt$ .
5. Let  $a < b$  be real numbers. Prove that the constant function  $\sqrt{\frac{1}{2}}$  plus the collection given by  $\{\cos(\frac{2\pi}{b-a} k(t - \frac{b+a}{2})), \sin(\frac{2\pi}{b-a} k(t - \frac{b+a}{2}))\}$  is an orthonormal basis for  $C[a, b]$  if the dot product is such as to assign any two functions  $f$  and  $g$  the number  $\frac{2}{(b-a)} \int_a^b f(t)g(t) dt$ .

## 10.4 Partial differential equations I: The heat/diffusion equation

There are significant applications of Fourier series in the theory of partial differential equations. In this regard, our discussion will focus mostly on the heat equation, but two other equations are also mentioned, Laplace's equation and the wave equation. This section studies the first of these.

The heat equation and the diffusion equation are one and the same, although they arise in different contexts. For the sake of simplicity, we call it the heat equation. Here it is:

**Definition 10.4.1:** *The heat or diffusion equation is for a function,  $T(t, x)$ , of time  $t$  and position  $x$ . The equation involves a positive constant,  $\mu$ , and has the form*

$$\frac{\partial T}{\partial t} = \mu \frac{\partial^2 T}{\partial x^2} .$$

As is plainly evident, the heat equation relates one time derivative of  $T$  to two spacial derivatives. A typical problem is one in where the interest is focused only on points  $x$  in some interval  $[a, b] \in \mathbb{R}$  with  $T$  some given function of  $x$  at time zero. The task then is to solve the heat equation for  $T(t, x)$  at times  $t > 0$  and points  $x \in [a, b]$ . Often, there are constraints imposed on  $T$  at the endpoints  $x = a$  and  $x = b$  that are meant to hold for all  $t$ .

Here is a sample problem: Take  $a = -\pi$  and  $b = \pi$  so that the focus is on values of  $x$  in  $[-\pi, \pi]$ . Suppose that we are told that  $T(0, x) = f(x)$  with  $f$  some given function of  $x$  for  $x \in [-\pi, \pi]$ . The task is to find the functional form of  $T$  at all times  $t > 0$ .

Before we pursue this problem, let me explain where this equation comes from. The preceding equation is known as the heat equation because it is used with great accuracy to predict the temperature of a long, but relatively thin rod as a function of time and position,  $x$ , along the rod. Thus  $T(t, x)$  is the temperature at time  $t$  and position  $x$ . The constant  $\mu$  that appears measures something of the thermal conductivity of the rod.

The theoretical underpinnings of this equation are based on our understanding of the temperature of a small section of the rod as measuring the average energy in the random motions of the constituent atoms. Heat 'flows' from a high temperature region to a low temperature because collisions between the constituent atoms tend to equalize their energy. In this regard, you most probably have observed that when a fast moving object strikes a slower one (for example in billiards), the faster one is almost always slowed by the collision while the slower one speeds up.

In any event, it is an experimental fact that a low energy region adjacent to a high energy one will tend to gain energy at the expense of the higher energy region. A simple way to model in a quantitative fashion is to postulate that the rate of flow of energy

across any given slice of the rod at any given time has the form  $-\mu \frac{\partial T}{\partial x}$  where  $\mu$  is a positive constant and where the derivative is evaluated at the  $x$ -coordinate of the slice and at the given value of  $t$ . Note that the minus sign here is dictated by the requirement that the flow of energy is from a high temperature region to a low temperature one.

Granted such a postulate, what follows is an argument for an equation that predicts the temperature as a function of time. Remembering that temperature measures the energy in the random motions of the particles, let us do some bookkeeping to keep track of the energy in a small width section,  $[x, x+\delta x]$ , of the rod. Here, I take  $\delta x > 0$  but very small. Think of  $T(t, x)\delta x$  as measuring the energy in this section of the rod. The time derivative of  $T(t, x)\delta x$  measures the net rate of energy coming into and leaving the section of rod. The net flow (positive or negative) of energy into our section of the bar is a sum of two terms: One is the flow across the left hand edge of the section. This contribution is equal to  $-\mu(\frac{\partial T}{\partial x})|_x$ . The other is the flow across the right hand edge, and this contribution is equal to  $+\mu(\frac{\partial T}{\partial x})|_{x+\delta x}$ . Note the appearance of the + sign since flow into our region across the right hand edge is flow in the direction that makes the rod's coordinate decrease.

Summing these two terms finds

$$\frac{\partial T}{\partial t}(t, x) \delta x = \mu \left( \frac{\partial T}{\partial x}(t, x + \delta x) - \frac{\partial T}{\partial x}(t, x) \right).$$

To end the derivation, divide both sides by  $\delta x$  and observe that

$$\frac{1}{\delta x} \left( \frac{\partial T}{\partial x}(t, x + \delta x) - \frac{\partial T}{\partial x}(t, x) \right) \approx \frac{\partial^2 T}{\partial x^2}(t, x)$$

when  $\delta x$  is very small. Thus, we see that

$$\frac{\partial T}{\partial t}(t, x) \approx \frac{\partial^2 T}{\partial x^2}(t, x)$$

when  $\delta x$  is very small.

Turn now to the task of solving the heat equation in Definition 10.4.1 for  $T(t, x)$  at values of  $t \geq 0$  and  $x \in [-\pi, \pi]$  given that  $T(0, x) = f(x)$ . To explain how this is done, introduce the space,  $C^\infty[-\pi, \pi]$ , of infinitely differentiable functions of  $x \in [-\pi, \pi]$  and then view the assignment

$$h(x) \rightarrow \frac{d^2 h}{dx^2}$$

as defining a linear operator on this space. The operator is linear because the second derivatives of a sum of functions is the sum of the second derivatives; and the second derivative of a constant times a function is equal to the same constant times the second derivative of the function. It is customary to call this linear operator the ‘Laplacian’ and denote it by  $\Delta$ . With this as notation, the heat equation then asks for a function  $T$  that obeys the equation  $\frac{\partial T}{\partial t} = \Delta T$ .

As noted in the introductory remarks, we dealt with equations of just this form in the case that  $T$  was a vector in  $\mathbb{R}^n$  and  $\Delta$  a linear operator from  $\mathbb{R}^n$  to itself. In the latter case, we were able to find explicit solutions when the linear operator on  $\mathbb{R}^n$  was diagonalizable. Let me remind you again of how this went: Granted that  $A$  is a diagonalizable linear operator on  $\mathbb{R}^n$ , let  $\{\bar{u}_1, \dots, \bar{u}_n\}$  denote its set of associated eigenvectors, a basis for  $\mathbb{R}^n$ . Each eigenvector has its associated eigenvalue, a real or complex number. The eigenvalue associated to  $\bar{u}_k$  is denoted here by  $\lambda_k$ . Now suppose that  $\bar{v}_0$  is a given vector in  $\mathbb{R}^n$  and suppose that we want to find the vector-valued function of time,  $t \rightarrow \bar{v}(t)$ , that obeys the equation  $\frac{d}{dt} \bar{v} = A \bar{v}$  subject to the constraint that  $\bar{v}(0) = \bar{v}_0$ . We do this by first writing  $\bar{v}_0$  in terms of the basis  $\{\bar{u}_k\}$  as  $\bar{v}_0 = \sum_k a_k \bar{u}_k$  with each  $a_k$  a scalar. This done, we then found that the solution can be written as

$$\bar{v}(t) = \sum_{1 \leq k \leq n} e^{\lambda_k t} a_k \bar{u}_k.$$

Our strategy for solving Definition 10.4.1’s heat equation for a function  $T(t, x)$  of  $x \in [-\pi, \pi]$  subject to the initial condition  $T(0, x) = f(x)$  is the infinite dimensional analog of that just described. Thus, the first step is to find a basis for the functions on  $[-\pi, \pi]$  that consists of eigenvectors of the linear operator  $\Delta$ . This might seem like a daunting task were it not for the seemingly serendipitous fact that every function in the Fourier basis

$$\left\{ \sqrt{\frac{1}{2}}, \cos(x), \sin(x), \cos(2x), \sin(2x), \cos(3x), \sin(3x), \dots \right\}$$

is an eigenfunction of  $\Delta$ . Indeed,  $\Delta \sqrt{\frac{1}{2}} = 0$  and for each  $k > 0$ , and

$$\frac{d^2}{dx^2} \cos(kx) = -k^2 \cos(kx) \quad \text{and} \quad \frac{d^2}{dx^2} \sin(kx) = -k^2 \sin(kx).$$

Thus, we have the following observation:

**Fact 10.4.2:** *Let  $f(x)$  denote any given continuous function on  $[-\pi, \pi]$  with continuous derivative, and write its Fourier series as*

$$f(x) = a_0 \sqrt{\frac{1}{2}} + \sum_{k=1}^{\infty} (a_k \cos(kx) + b_k \sin(kx)) .$$

Then there is a function,  $T(t, x)$ , that is defined for  $t \geq 0$  and  $x \in [-\pi, \pi]$ ; that solves the heat equation for  $t > 0$  with initial condition  $T(0, x) = f(x)$  for all  $x \in (-\pi, \pi)$ ; and whose Fourier series with respect to the variable  $x$  at any given  $t \geq 0$  is

$$T(t, x) = a_0 \sqrt{\frac{1}{2}} + \sum_{k=1}^{\infty} (a_k e^{-\mu k^2 t} \cos(kx) + b_k e^{-\mu k^2 t} \sin(kx)) .$$

If it is also the case that  $f(\pi) = f(-\pi)$ , then  $T(t, \pi) = T(t, -\pi)$  for all  $t \geq 0$  and these are equal to  $f(\pi)$  at  $t = 0$ .

Here is a first example: Suppose, that  $f(x) = (\pi^2 - x^2)$ . We found its Fourier series in the previous part of this chapter,

$$f(x) = \frac{2}{3} \pi^2 - 4 \sum_{k=1}^{\infty} (-1)^k \frac{1}{k^2} \cos(kx) .$$

In this case, the function  $T(t, x)$  given by Fact 10.3.2 is

$$T(t, x) = \frac{2}{3} \pi^2 - 4 \sum_{k=1}^{\infty} (-1)^k \frac{1}{k^2} e^{-\mu k^2 t} \cos(kx) .$$

Here is a second example: Take  $f(x) = e^x$ . From one of the examples in the previous part of this chapter, we see that the function  $T(t, x)$  that is given by Fact 10.3.2 in this case is

$$T(t, x) = \frac{1}{\pi} (e^{\pi} - e^{-\pi}) \left[ \frac{1}{2} + \sum_{k=1}^{\infty} (-1)^k e^{-\mu k^2 t} \left( \frac{1}{1+k^2} \cos(kx) - \frac{k}{1+k^2} \sin(kx) \right) \right] .$$

In all fairness, I should point out that there is some tricky business here that doesn't arise in the finite dimensional model problem  $\frac{d}{dt} \vec{v} = A \vec{v}$ . In particular, there are non-zero solutions to the heat equation for  $x \in [-\pi, \pi]$  whose time zero solution is the constant function  $f(x) \equiv 0$  for all  $x$ ! The existence of such solutions is the manifestation of some facts about heat and diffusion that I haven't mentioned but surely won't surprise you if you have lived in a drafty old house: The distribution of heat in a room is not completely determined by the heat at time zero because you must take into account the heat that enters and leaves through the walls of the room. Thus, in order to completely pin down a unique solution to the heat equation, the function of  $x$  given by  $T(0, x)$  must be specified—this corresponds to the heat distribution at time zero in our hypothetical rod—but the functions  $T(t, \pi)$  and  $T(t, -\pi)$  of time must also be specified so as to pin down the amount of heat that enters and leaves the ends of our hypothetical rod.

What follows are some examples of solutions that are non-zero for  $t > 0$  on the interval where  $x$  is between  $-\pi$  and  $\pi$ , but are zero at all points in this interval when  $t = 0$ : Choose any point,  $a$ , that is not in the interval  $[-\pi, \pi]$  and the function

$$T(t, x) = \frac{1}{\sqrt{t}} e^{-(x-a)^2/(2\mu t)}$$

solves the heat equation for  $t > 0$ . Moreover, in spite of the factor  $\frac{1}{\sqrt{t}}$ , its  $t \rightarrow 0$  limit at points  $x \in [-\pi, \pi]$  is zero. Here is where the condition  $a \notin [-\pi, \pi]$  is crucial. To say more, note that the factor  $(x-a)^2/2\mu t$  blows up as  $t \rightarrow 0$  if  $x \neq a$ . In particular, its negative exponential is tiny and converges to zero as  $t \rightarrow 0$ . This convergence is much faster than the rate of blow up of  $\frac{1}{\sqrt{t}}$ . To see that this is indeed the case, consider that the time derivative of  $\frac{1}{t} T(t, x)$  is

$$\left(-\frac{3}{2} \frac{1}{t} + \frac{(x-a)^2}{2\mu t^2}\right) \frac{1}{t} T(t, x)$$

which is positive when

$$t < \frac{(x-a)^2}{3\mu} .$$

Thus,  $\frac{1}{t} T(t, x)$  is increasing for small  $t$  as long as  $x \neq a$ . Since  $\frac{1}{t} T(t, x)$  is not negative, it must have a limit as  $t \rightarrow 0$  from the positive side since it decreases with  $t$  decreasing given that  $t$  is small. Let's call this limit  $c$ . Then  $T(t, x) \sim t c$  for small  $t$ , and this implies that  $T(0, x) = 0$ . (As it turns out,  $T(t, x)$  goes to zero as  $t \rightarrow 0$  faster than any given power of  $t$ .)

Any constraint on the boundary values,  $T(t, \pi)$  and  $T(t, -\pi)$ , for  $T$  at all times  $t > 0$  is called a *boundary condition* for  $T$ . For example, I would require that my solution  $T$  obey  $T(t, \pi) = 20$  for all  $t$  were I to keep the  $x = \pi$  end of the rod at 20 degrees temperature. The existence of solutions to the heat equation with prescribed boundary conditions is an important subject, but one that we won't pursue in this course.

### Exercises

1. Solve the heat equation for a function  $T(t, x)$  of  $t \geq 0$  and  $x \in [-\pi, \pi]$  that obeys the initial condition  $T(0, x) = \sin^2(x) - \cos^4(x)$ . (Rather than do the integrals for the Fourier series, take the following shortcut: Use standard trigonometric identities to write  $T(0, x)$  as a sum of sine and cosine functions.)
2. Use Fourier series to solve the heat equation for a function  $T(t, x)$  of  $t \geq 0$  and

$x \in [-\pi, \pi]$  that obeys the initial condition  $T(0, x) = \sinh(x)$ . You can avoid many of the integrals by exploiting the Fourier series solution for the initial condition  $e^x$  given above.

3. Suppose that  $c$  is a constant. Prove that  $T(t, x) = e^{uc^2t} e^{cx}$  solves the heat equation.
4. Take the case  $c = 1$  in the previous problem and prove that the resulting solution of the heat equation with the initial condition  $T(0, x) = e^x$  is not the same as the one given in the text, above. (Hint: Compare the corresponding Fourier series.)
5. Use Fourier series to solve the heat equation for a function  $T(t, x)$  for  $t \geq 0$  and for  $x \in [-\pi, \pi]$  subject to the initial condition  $T(0, x) = x$ .
6. Prove that  $T(t, x) = x$  is also a solution to the heat equation for  $t \geq 0$  and  $x \in [-\pi, \pi]$  with the initial condition  $T(0, x) = x$ . Prove that it is different than the one you found in the Problem 5 using Fourier series.

## 10.5 Partial differential equations II: Fourier series and the Laplace equation

Fourier series can be used to find solutions to many other sorts of differential equations. The discussion that follows considers Fourier series in the context of the Laplace equation and the wave equation.

The Laplace equation is for a function,  $T$ , of two space variables,  $x$  and  $y$ . Here is the definition:

**Definition 10.5.1:** *A function  $u$  that is defined on some given region in the  $x$ - $y$  plane is said to obey the Laplace equation in the case that*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

*at all points  $(x, y)$  in the given region.*

Versions of this equation arise in numerous areas in the sciences. Those of you who plan to take a course about electricity and magnetism will see it. Likewise, if you study the analog of the heat/diffusion equation for a thin plate whose shape is given by a region in the  $x$ - $y$  plane, you will see that time independent solutions to the heat/diffusion equation are solutions to the Laplace equation. Indeed, this is because the two dimensional version of the heat equation is for a function  $T(t, x, y)$  of time and the space coordinates  $x$  and  $y$  that obeys the equation

$$\frac{\partial T}{\partial t} = \mu \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right).$$

If  $T$  is an equilibrium solution to this last equation, then it depends only on the space coordinates  $x$  and  $y$  and so supplies a solution to the Laplace equation.

Here is a basic fact about the Laplace equation and its solutions:

*Suppose that  $R$  is a bounded region in the  $x$ - $y$  plane whose boundary is a finite union of segments of smooth curves. Suppose in addition that  $f$  is a continuous function that is defined on the boundary of  $R$ . Then there is a unique solution to the Laplace equation inside  $R$  whose restriction to the boundary is equal to  $f$ .*

To explain some of the terminology, a segment of a smooth curve means simply a part of a level set of some function,  $h(x, y)$ , where the gradient of  $h$  is nonzero. Here,  $h$  is assumed to have partial derivatives to all orders with respect to the variables  $x$  and  $y$ . Thus, a smooth curve is part of the locus of points  $h(x, y) = c$ , where  $c$  is a specified constant. But, it must lie in the portion of this level set where at least one of  $\frac{\partial}{\partial x} h$  and  $\frac{\partial}{\partial y} h$

is not zero. For example, except for (0, 0), the locus where  $x^3 + xy^2 = 0$  is a smooth curve.

If you haven't yet taken a multivariable calculus course, this explanation and the constraints on the region most probably seem like mumbo-jumbo. If so, don't fret because the discussion that follows concentrates exclusively on the case where the region  $R$  is the square where  $-\pi \leq x \leq \pi$  and  $-\pi \leq y \leq \pi$ . With this proviso understood, here is a formal restatement of what was just said:

**Fact 10.5.2:** *Consider the square where both  $-\pi \leq x \leq \pi$  and  $-\pi \leq y \leq \pi$ . Suppose that  $f$  is a continuous function that is defined on the boundary of the square. There is a unique solution to the Laplace equation in the square that has partial derivatives to all orders at points inside the square and whose restriction to the boundary is equal to  $f$ .*

The facts about Fourier series from Section 10.3 can be used to obtain the predicted solution in certain cases. I describe below how this works when the function  $f$  is of the following sort: It is zero on the parts of the boundary where  $x = \pi$  and  $x = -\pi$ , and its Fourier series expansions on the respective  $y = -\pi$  and  $y = \pi$  boundaries contain only sine functions. Arguments for Fact 10.5.2 when  $f$  does not have these properties can be made using a Fourier series expansion that is somewhat different than that presented in presented Section 10.3. In any event, the general case of Fact 10.5.2 is not discussed below.

The discussion of Fact 10.5.2 begins here with a digression to describe a vector analog of the Laplace equation that I use as a guide. To start the digression, suppose that  $A$  is a diagonalizable,  $n \times n$  matrix, and that I am looking for a vector in  $\mathbb{R}^n$  that depends on the variable  $y$  for values of  $y$  between  $-\pi$  and  $\pi$ , and that solves the equation

$$\frac{d^2}{dy^2} \vec{v} = A \vec{v}.$$

The analogy with the Laplace equation comes by writing the Laplace equation as

$$\frac{d^2}{dy^2} T = -\frac{d^2}{dx^2} T,$$

and thinking of  $T$  as a  $y$ -dependent vector in the vector space whose elements are functions of  $x$  for  $x$  between  $-\pi$  and  $\pi$ . With  $T$  playing the role of  $\vec{v}$ , then the role of  $A$  is played by the linear transformation

$$h \rightarrow -\frac{d^2 h}{dx^2} .$$

With this analogy in mind, here is how I would solve the equation for  $\bar{v}(y)$ : I first introduce a basis,  $\{\bar{u}_1, \dots, \bar{u}_n\}$  of eigenvectors for the matrix  $A$ . Thus,  $A\bar{u}_k = \lambda_k \bar{u}_k$ , where  $\lambda_k$  is the corresponding eigenvalue. I then write

$$\bar{v}(y) = a_1(y) \bar{u}_1 + a_2(y) \bar{u}_2 + \dots + a_n(y) \bar{u}_n$$

which I can do for each value of  $y$  since is a basis for  $\mathbb{R}^n$ . With  $\bar{v}(y)$  written as above, I differentiate it twice with respect to  $y$  and see that the differential equation for  $\bar{v}$  is satisfied if and only if each  $a_k(y)$  obeys the equation

$$\frac{d^2}{dy^2} a_k(y) = \lambda_k a_k(y) .$$

This last equation is one of the sort that were analyzed in Section 10.3. Let me remind you that the solutions are of the following form when  $\lambda_k > 0$ :

$$a_k(y) = \alpha_k e^{r_k y} + \beta_k e^{-r_k y} ,$$

where  $r_k$  is the square root of  $\lambda_k$  and where  $\alpha_k$  and  $\beta_k$  are constants. As it turns out, the case  $\lambda_k > 0$  is the case that is relevant to the ensuing discussion of Laplace's equation.

Granted the preceding, I find that the general solution to my vector analog of the Laplace equation has the form

$$\bar{v}(y) = (\alpha_1 e^{r_1 y} + \beta_1 e^{-r_1 y}) \bar{u}_1 + \dots + (\alpha_n e^{r_n y} + \beta_n e^{-r_n y}) \bar{u}_n .$$

With this vector model understood, turn now to the Laplace equation. Suppose that I wish to find a solution,  $T(x, y)$ , in the square that has value zero on both the  $x = -\pi$  and  $x = \pi$  boundaries, but has some prescribed behavior where  $y = -\pi$  and where  $y = \pi$ . To be specific about this last point, I will assume that functions  $x \rightarrow f(x)$  and  $x \rightarrow g(x)$  are given to me, both defined for values of  $x$  between  $-\pi$  and  $\pi$ . Moreover, both have only sine functions in their Fourier series. My solution to the Laplace equation should obey  $T(x, -\pi) = f(x)$  and  $T(x, \pi) = g(x)$ .

For example, I ask that you imagine looking for a solution in the square with the property that it is zero where  $x = -\pi$  and  $x = +\pi$ , equals  $\sin(x)$  where  $y = -\pi$  and  $\sin(2x)$  where  $y = +\pi$ . Or, imagine looking for a solution that is zero where  $x = -\pi$  and  $x = \pi$ , and equals  $x(\pi^2 - x^2)$  where  $y = -\pi$  and equals 0 where  $y = \pi$ .

To find the predicted solution, I invoke the vector space analogy from the preceding digression and so start by viewing the function  $T(y, x)$  as a  $y$ -dependent vector in the vector space of smooth functions of  $x$  for  $-\pi \leq x \leq \pi$ . I then recall from Section 10.4 that the set of functions  $\{\sqrt{\frac{1}{2}}, \cos(x), \sin(x), \cos(2x), \sin(2x), \cos(3x), \sin(3x), \dots\}$  are all eigenvectors for the transformation

$$h \rightarrow -\frac{d^2h}{dx^2}.$$

As both  $f(x)$  and  $g(x)$  contain only sine functions in their Fourier series, I use only the eigenvectors from the set  $\{\sin(x), \sin(2x), \dots\}$ . In this regard, note that the eigenvalue of  $\sin(kx)$  for the transformation above is  $k^2$ .

With the preceding understood, I now copy what I did in the vector analog from the opening digression to write down the general solution,  $T(x, y)$ , to the Laplace equation on the square with  $T(-\pi, y) = 0$  and  $T(\pi, y) = 0$  for all  $y$ , and with both  $T(x, -\pi)$  and  $T(x, \pi)$  containing only sine functions in their Fourier series expansion:

$$T(x, y) = (\alpha_1 e^y + \beta_1 e^{-y}) \sin(x) + (\alpha_2 e^{2y} + \beta_2 e^{-2y}) \sin(2x) \cdots + (\alpha_k e^{ky} + \beta_k e^{-ky}) \sin(kx) + \cdots.$$

Here, each  $\alpha_k$  and  $\beta_k$  is a constant. Note that there are a infinitely many solutions because one can freely specify  $\alpha_1, \beta_1, \alpha_2, \beta_2, \dots$  etc. As I explain next, this freedom with the constants allows us to specify what  $T$  should be where  $y = \pi$  and  $y = -\pi$ .

To see how this freedom is used, consider an example: Suppose that I want a solution that vanishes where  $x = \pi$  and  $x = -\pi$  with  $T(x, -\pi) = 0$  and  $T(x, \pi) = \sin(x)$ . As I look at the general solution, I see that these requirements can be met if  $\alpha_k = 0$  and  $\beta_k = 0$  for  $k \geq 2$ , and if

$$\alpha_1 e^{-\pi} + \beta_1 e^{\pi} = 0 \quad \text{and} \quad \alpha_1 e^{\pi} + \beta_1 e^{-\pi} = 1.$$

Thus, I have the desired solution if I can solve this system of linear equations.

As you are all now experts on such linear systems, you have surely noted that it is solvable if the matrix

$$M_1 = \begin{pmatrix} e^{-\pi} & e^{\pi} \\ e^{\pi} & e^{-\pi} \end{pmatrix}$$

is invertible. Moreover, if  $M$  is invertible, you know that there is a unique solution. Since the determinant of this matrix,  $e^{-2\pi} - e^{2\pi}$ , is non-zero, there, in fact, a unique solution to the desired system of equations:

$$\alpha_1 = \frac{e^\pi}{e^{2\pi} - e^{-2\pi}} \quad \text{and} \quad \beta_1 = -\frac{e^{-\pi}}{e^{2\pi} - e^{-2\pi}} .$$

Thus, the sought after solution of Laplace's equation is

$$\left( \frac{e^{\pi+y}}{e^{2\pi} - e^{-2\pi}} - \frac{e^{-\pi-y}}{e^{2\pi} - e^{-2\pi}} \right) \sin(x).$$

Here is another example: Suppose that  $c$  and  $c'$  are constants and I want the solution to the Laplace equation on the square that vanishes where  $x = \pi$  and  $x = -\pi$ , equals  $c \sin(x)$  where  $y = -\pi$  and equals  $c' \sin(x)$  where  $y = \pi$ . I again take all  $k \geq 2$  versions of  $\alpha_k$  and  $\beta_k$  equal to zero and take  $\alpha_1$  and  $\beta_1$  to be solutions to the linear system

$$\alpha_1 e^{-\pi} + \beta_1 e^\pi = c \quad \text{and} \quad \alpha_1 e^\pi + \beta_1 e^{-\pi} = c' .$$

As this system involves the matrix  $M_1$  again, I know there is a unique solution:

$$\alpha_1 = \frac{1}{e^{2\pi} - e^{-2\pi}} (-e^{-\pi} c + e^\pi c') \quad \text{and} \quad \beta_1 = \frac{1}{e^{2\pi} - e^{-2\pi}} (e^\pi c - e^{-\pi} c') .$$

Thus, the desired solution to the Laplace equation is

$$\frac{1}{e^{2\pi} - e^{-2\pi}} ((-e^{-\pi} c + e^\pi c') e^y + (e^\pi c - e^{-\pi} c') e^{-y}) \sin(x) .$$

To continue with examples, again take  $c$  and  $c'$  to be constants, and now let  $n$  denote a positive integer. Suppose that I want the solution,  $T(x, y)$ , to the Laplace equation on the square that is zero where  $x = \pi$  and  $x = -\pi$ , equals the function  $c \sin(nx)$  where  $y = -\pi$ , and equals the function  $c' \sin(nx)$  where  $y = \pi$ . This solution will have all  $\alpha_k$  and  $\beta_k$  equal zero except  $\alpha_n$  and  $\beta_n$ . Meanwhile,  $\alpha_n$  and  $\beta_n$  must solve the linear system

$$\alpha_n e^{-n\pi} + \beta_n e^{n\pi} = c \quad \text{and} \quad \alpha_n e^{n\pi} + \beta_n e^{-n\pi} = c' .$$

This system involves the matrix

$$M_n = \begin{pmatrix} e^{-n\pi} & e^{n\pi} \\ e^{n\pi} & e^{-n\pi} \end{pmatrix}$$

which is invertible since its determinant is  $e^{-2n\pi} - e^{2n\pi}$ . This understood, I can write down the solution to the linear system:

$$\alpha_1 = \frac{1}{e^{2n\pi} - e^{-2n\pi}} (-e^{-n\pi} c + e^{n\pi} c') \quad \text{and} \quad \beta_1 = \frac{1}{e^{2n\pi} - e^{-2n\pi}} (e^{n\pi} c - e^{-n\pi} c').$$

Thus, the desired solution to the Laplace equation is

$$\frac{1}{e^{2n\pi} - e^{-2n\pi}} ((-e^{-n\pi} c + e^{n\pi} c') e^{ny} + (e^{n\pi} c - e^{-n\pi} c') e^{-ny}) \sin(nx).$$

Having understood these basic examples, turn now to the case where the solution,  $T(x, y)$ , to the Laplace equation is sought with the following properties: First,  $T(-\pi, y)$  and  $T(\pi, y)$  are both zero for all  $y$ . Second,  $T(x, -\pi) = f(x)$  and  $T(x, \pi) = g(x)$  where  $f$  and  $g$  are given functions that are defined for values of  $x$  between  $-\pi$  and  $\pi$  and have only sine functions in their Fourier series expansions. To make this last point explicit, I am assuming that

$$f(x) = \sum_{n=1,2,\dots} c_n \sin(nx) \quad \text{and} \quad g(x) = \sum_{n=1,2,\dots} c_n' \sin(nx)$$

where each  $c_n$  and  $c_n'$  is a constant.

Now, you have just found the solution in the case where all but one pair of the numbers  $(c_n, c_n')$  are zero. Just take  $c = c_n$  and  $c' = c_n'$  in the formula one paragraph back. To get the solution in general, simply add all of these different  $n$  versions together. This is to say that the desired  $T(x, y)$  is

$$T(x, y) = \sum_{n=1,2,\dots} \frac{1}{e^{2n\pi} - e^{-2n\pi}} ((-e^{-n\pi} c_n + e^{n\pi} c_n') e^{ny} + (e^{n\pi} c_n - e^{-n\pi} c_n') e^{-ny}) \sin(nx).$$

Note that in writing  $T(x, y)$  as above, I am invoking a fundamental fact about the Laplace equation:

**Fact 10.5.3:** *Suppose that  $T$  and  $T'$  are both solutions to the Laplace equation that are defined in the same region of the  $x$ - $y$  plane. Then  $T + T'$  is also a solution to the Laplace equation on the given region. In fact, if  $r$  and  $r'$  are any pair of real numbers, then the function  $rT + r'T'$  is a solution to the Laplace equation on the given region.*

An equation is called linear when it has the property that new solutions result by constant multiples of old ones. Fact 10.5.3 asserts that the Laplace equation is a linear equation.

The Laplace equation is linear because the act of taking a partial derivative is linear. This is to say that the partial derivative of a sum of functions is the sum of their partial derivatives. Likewise, the partial derivative of a constant times a function is just the constant times the partial derivative of the function.

### Exercises

1. This exercise concerns the square in the  $x$ - $y$  plane where  $-\pi \leq x \leq \pi$  and  $-\pi \leq y \leq \pi$ . Suppose that the  $x = -\pi$ ,  $x = \pi$ , and  $y = -\pi$  boundaries of the square are kept at 0 degrees centigrade. Suppose, in addition, that the temperature of the  $y = \pi$  boundary depends on the position  $x$  with the temperature at  $x$  equal to  $3\sin(3x) - 2\sin(7x)$ .
  - a) What is the solution to the Laplace equation on the square with these boundary conditions?
  - b) What is the temperature of the square at the point  $(\frac{\pi}{2}, \frac{\pi}{2})$  if the temperature does not change with time?
  
2. This exercise concerns the same square in the  $x$ - $y$  plane, that where both  $x$  and  $y$  range only between  $-\pi$  and  $\pi$ . Suppose that the temperature of both the  $x = -\pi$  and  $x = \pi$  boundaries are kept at 0 degrees centigrade, that the temperature of the  $y = -\pi$  boundary at any given point  $x$  is  $-\sin(5x) + 12 \sin(6x)$ , and that the temperature of the  $y = \pi$  boundary at any given point  $x$  is  $-\sin(5x)$ .
  - a) What is the solution to the Laplace equation on the square with these boundary conditions?
  - b) What is the temperature in the square at the point  $(\frac{\pi}{2}, 0)$  if the temperature is not changing with time?
  
3. This exercise also concerns the square in the  $x$ - $y$  plane where both  $x$  and  $y$  range only between  $-\pi$  and  $\pi$ . However, note that the roles of  $x$  and  $y$  are switched! Suppose that the temperature of both the  $y = -\pi$  and  $y = \pi$  boundaries are kept at 0 degrees centigrade, that the temperature of the  $x = -\pi$  boundary at any given  $y$  is  $\sin(3y)$ , and that the temperature of the  $x = \pi$  boundary at any given  $y$  is  $\sin(y) + 12 \sin(3y)$ .
  - a) What is the solution to the Laplace equation on the square with these boundary conditions?
  - b) What is the temperature in the square at the point  $(0, \frac{\pi}{2})$  if the temperature is not changing with time?
  
4. Suppose that  $T(x, y)$  is a solution to the Laplace equation on the square with the following properties: First  $T(-\pi, y) = 0$  and  $T(\pi, y) = 0$  for all  $y$ . Second,  $T(x, -\pi) = \sin(x)$ , and  $T(x, \pi) = 0$  for all  $x$ . Let  $T'(x, y)$  denote a second solution to Laplace's

equation on the square, but this solution obeys  $T'(x, -\pi) = 0$  and  $T'(x, \pi) = 0$  for all  $x$ ,  $T(-\pi, y) = \sin(2y)$ , and  $T(\pi, y) = 0$  for all  $y$ .

- Write down the function  $T$ .
- Write down the function  $T'$ .
- What is the value at the point  $(\frac{\pi}{2}, \frac{\pi}{2})$  of the Laplace equation on the square with the boundary conditions  $T(-\pi, y) = \sin(2y)$ ,  $T(\pi, y) = 0$ ,  $T(x, -\pi) = \sin(x)$  and  $T(x, \pi) = 0$ ?

### 10.6 Partial differential equations III: Other equations

Many often used differential equations in the sciences can be analyzed with analogs of our method solving the equation

$$\frac{d}{dt} \vec{v} = A \vec{v}$$

for a time dependent vector,  $\vec{v}$ , in  $\mathbb{R}^n$  in the case where  $A$  is a diagonalizable,  $n \times n$  matrix. I expect that you now know write the general solution to this last equation as

$$\vec{v}(t) = \sum_{1 \leq k \leq n} e^{\lambda_k t} a_k \vec{u}_k,$$

where  $\{\vec{u}_k\}_{1 \leq k \leq n}$  are a basis of eigenvectors for the matrix  $A$ , the real or complex numbers  $\{\lambda_k\}_{1 \leq k \leq n}$  are the corresponding set of eigenvalues, and each  $a_k$  is a constant.

One differential equation to which the analogy applies is the *Schrödinger equation*. The latter is the key equation of quantum physics and quantum chemistry. A simple version is for a complex valued function of time  $t$  and position,  $x$ . It is traditional to denote this function as  $\psi$ . The function  $\psi$  solves the simplest version of the Schrödinger equation when

$$i \frac{\partial}{\partial t} \psi = -\kappa \frac{\partial^2}{\partial x^2} \psi,$$

where  $\kappa$  is a positive constant that is determined by the particulars of the problem under consideration. The analogy with the vector equation can be seen after multiplying both sides of this equation by  $-i$ . Having done this, view  $\psi$  as a time dependent vector in a vector space of functions of  $x$  so it can play the role of  $\vec{v}$  in the  $\mathbb{R}^n$  case. Here, you should think of the vector space of functions on  $\mathbb{R}$  or some sub-interval of  $\mathbb{R}$  that can be differentiated as many times as desired. The transformation that sends a function,  $\eta$ , of  $x$  with complex number values to

$$i\kappa \frac{\partial^2}{\partial x^2} \eta$$

plays the role of the linear transformation on  $\mathbb{R}^n$  that sends  $\vec{v}$  to  $A\vec{v}$ .

Another physics analog of our vector differential equation are *Maxwell's equations*. These equations describe the time and space variation of electric and magnetic fields. The electric field for Maxwell's equations is a vector in  $\mathbb{R}^3$  that depends on time  $t$ , and also on a point,  $(x, y, z)$ , in  $\mathbb{R}^3$ . It is traditional to write this time and space dependent vector as  $\vec{E}(t, x, y, z)$ . Likewise, the magnetic field is written as a time and space dependent vector,  $\vec{B}(t, x, y, z)$ . Maxwell's equations demand that

$$\frac{\partial}{\partial t} \vec{E} = -\frac{1}{c} \text{curl}(\vec{B}) \quad \text{and} \quad \frac{\partial}{\partial t} \vec{B} = \frac{1}{c} \text{curl}(\vec{E}).$$

Here,  $c$  denotes the speed of light. The curl of a vector field is something that those of you from Math 21a have seen. Those who do not know about the curl need only know the following: It takes a vector function of the coordinates  $(x, y, z)$  and spits out another. Moreover, it does this in a linear fashion in the sense that

$$\text{curl}(r\vec{a}) = r \text{curl}(\vec{a}), \quad \text{and} \quad \text{curl}(\vec{a}_1 + \vec{a}_2) = \text{curl}(\vec{a}_1) + \text{curl}(\vec{a}_2)$$

when  $r$  is a real number and  $\vec{a}$ ,  $\vec{a}_1$  and  $\vec{a}_2$  are vector functions of the coordinates  $(x, y, z)$ .

Except for this linearity property, the precise nature of the curl is not central to what is said next. The important point is that this linearity allows the Maxwell equations to be viewed as an analog of our vector equation. Here, the pair  $(\vec{E}, \vec{B})$  plays the role of the vector  $\vec{v}$ , and the transformation

$$(\vec{E}, \vec{B}) \rightarrow \left(-\frac{1}{c} \text{curl}(\vec{B}), \frac{1}{c} \text{curl}(\vec{E})\right)$$

plays the role of the  $n \times n$  matrix  $A$ .

The vector equation

$$\frac{d^2}{dt^2} \vec{v} = A\vec{v}$$

can also be used to solve a number of differential equations. We saw in Section 10.5 that the general solution to this last equation has the form

$$\bar{v}(t) = \sum_{1 \leq k \leq n} (e^{r_k t} \alpha_k + e^{-r_k t} \beta_k) \bar{u}_k ,$$

where  $r_k^2 = \lambda_k$ , and where  $\alpha_k$  and  $\beta_k$  are complex numbers. In this regard, keep in mind that  $r_k$  is a real number only when the eigenvalue  $\lambda_k$  is a non-negative, real number. Otherwise,  $r_k$  is a complex number.

The wave equation is one of the most ubiquitous of the differential equations that are analogous to this last equation for  $\bar{v}$ . The simplest version of the wave equation looks for a function,  $u(t, x)$ , that is defined for all values of  $t \in \mathbb{R}$  and for values of  $x$  that range over some interval  $[a, b]$ . Here is the definition of the equation for  $u$ :

**Definition 10.6.1:** *Suppose that a positive number,  $c$ , and numbers  $a < b$  have been specified. A function,  $u$ , of the variables  $t$  and  $x$  where  $t \in \mathbb{R}$  and  $x \in (a, b)$  is said to obey the wave equation in the case that*

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 .$$

at all of values of  $t \in \mathbb{R}$  and  $x \in (a, b)$ .

The wave equation is typically augmented with *boundary conditions* for  $u$  at the points where  $x = a$  and  $x = b$ . To keep the story short, I will only discuss the case where  $u$  is constrained so that

$$u(t, a) = 0 \text{ and } u(t, b) = 0 \text{ for all } t.$$

It is often the case that one must find a solution to the wave equation subject to additional conditions that constrain the value of  $u$  and its time derivative at  $t = 0$ . These are typically of the following form: Functions  $f(x)$  and  $g(x)$  on  $[a, b]$  are given that both vanish at the endpoints. A solution  $u(t, x)$  is then sought for the wave equation subject to the boundary conditions  $u(t, a) = 0 = u(t, b)$  and to the *initial conditions*

$$u(0, x) = f(x) \text{ and } \frac{\partial u}{\partial t}(0, x) = g(x) \text{ for all } x \in [a, b].$$

The equation in Definition 10.6.2 is called the *wave equation* because it is used to model the wave-like displacements (up/down) that are seen in vibrating strings. In this regard, such a model ignores gravity, friction and compressional effects as it postulates an idealized, tensed string whose equilibrium configuration stretches along the  $x$ -axis from where  $x = a$  to  $x = b$ , and whose ends are fixed during the vibration. The constant  $c$

that appears in the wave equation determines the fundamental frequency of the vibration,  $\frac{b-a}{\pi c}$ .

To elaborate,  $u(t, x)$  gives the  $z$ -coordinate of the string at time  $t$  over the point  $x$  on the  $x$ -axis. The boundary conditions  $u(t, a) = 0 = u(t, b)$  keeps the ends of the string fixed during the vibration. The initial conditions are specifying the state of the string at time 0. For example, in the case that  $g \equiv 0$ , the string is started at time zero at rest, but with a displacement at any given  $x$  equal to  $f(x)$ . As it turns out, such an idealization is quite accurate for small displacements in tautly stretched real strings. For example, the behavior of violin and other musical instrument strings are well described by the wave equation.

Somewhat more complicated versions of the wave equation are also used to model the propagation of sound waves, water waves, and sundry other wave-like phenomena.

The following summarizes what can be said about the existence of solutions:

**Fact 10.6.2:** *Let  $f(x)$  and  $g(x)$  be any two given functions that are defined on an interval where  $a \leq x \leq b$  that are zero at the endpoints. Assume that both can be differentiated arbitrarily many times. Then, there is a unique function,  $u(t, x)$ , that is defined for all  $t$  and for  $x \in [a, b]$ , and has the following properties:*

- $u(t, x)$  obeys the wave equation for all  $t$  and for all points  $x$  with  $a < x < b$ .
- $u(t, a) = u(t, b) = 0$  for all  $t$ .
- $u(0, x) = f(x)$  and  $\frac{\partial u}{\partial t}(0, x) = g(x)$  for all  $x \in [a, b]$ .

This is not the place to describe a proof of this fact. I only want to point out that a proof can be had by using our vector equation analogy. The role played by  $\vec{v}$  in the vector equation is taken by  $u$  when the latter is viewed as a time dependent vector in the vector space of functions that are defined where  $a \leq x \leq b$  and that can be differentiated to any desired order. The role played by the  $n \times n$  matrix  $A$  is played here by the transformation that takes a function,  $h$ , that is defined for  $a \leq x \leq b$ , to the function

$$c^2 \frac{\partial^2 h}{\partial x^2}.$$

Here is the most important lesson to take away from Chapter 10, and from this last part of the chapter in particular:

*Many problems involving functions and differential equations can be solved by using analogies to problems that involve vectors and matrices on  $\mathbb{R}^n$ .*

